University of Kentucky, Physics 420
Homework #3, Rev. A, due Monday, 2015-09-28

1. Given the **non-orthogonal basis** $\vec{b}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\vec{b}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$,

a) Write the equation $\vec{v} = \vec{b}_1 v^1 + \vec{b}_2 v^2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ in matrix form and use it to solve for the components $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ of $\vec{v}$ in the basis $(\, \vec{b}_1 \quad \vec{b}_2 \,)$.

b) Calculate the dual vectors $\tilde{b}^1 = (\, b^1_x \quad b^1_y \,)$ and $\tilde{b}^2 = (\, b^2_x \quad b^2_y \,)$ so that $\tilde{b}^i(\vec{b}_j) = \delta^i{}_j$, ie.

$$\begin{aligned} \tilde{b}^1(\vec{b}_1) &= 1 \qquad \tilde{b}^1(\vec{b}_2) = 0 \\ \tilde{b}^2(\vec{b}_1) &= 0 \qquad \tilde{b}^2(\vec{b}_2) = 1. \end{aligned}$$

What is the relation between the basis and cobasis matrices $\vec{B} = (\, \vec{b}_1 \quad \vec{b}_2 \,)$ and $\tilde{B} = \begin{pmatrix} \tilde{b}^1 \\ \tilde{b}^2 \end{pmatrix}$? Note the appearance of these matrices in part a). Use these matrices to resolve the components in $\vec{B}$ of an arbitrary vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ or the components in $\tilde{B}$ of an arbitrary dual $\tilde{w} = (\, w_1 \quad w_2 \,)$.

c) Calculate the two projector (matrices) $\boldsymbol{P}_i = \vec{b}_i \tilde{b}^i$ and show that they project $\vec{v}$ onto each of the basis vectors $\vec{b}_i$. Add them to show that $\sum_{i=1}^{2} \boldsymbol{P}_i = I$ (closure). How does that follow from part b)?

d) Calculate the metric $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ so that $\vec{x} \cdot \vec{y} = (\, x^1 \quad x^2 \,) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$. Verify this formula on the components of $\vec{v} \cdot \vec{v}$ in both the normal and $\vec{B}$ basis. Note that $g_{ij} = \vec{b}_i \cdot \vec{b}_j$.

e) [bonus] In this non-orthonormal basis, write the formula for the adjoint $\tilde{v} = (\, v_1 v_2 \,)$ of $\vec{v}$, defined by $\tilde{v}(\vec{x}) = \vec{v} \cdot \vec{x}$ for any other vector $\vec{x}$.

2. Circular polarized photons travelling in the z-direction may be described by the wave function $\Psi(\vec{r}) = \hat{\boldsymbol{x}} e^{ikz - i\omega t} + i\hat{\boldsymbol{y}} e^{ikz - i\omega t}$.

a) What is physical difference between the x and y components? Plot the real part of this wave.

b) Use the Gram-Schmidt method to normalize $\hat{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and complete the basis with a second orthonormal vector $\hat{\mathbf{e}}_2$. Note that in this context our scalar field is $\mathbb{C}$

c) Calculate the adjoints $\hat{\mathbf{e}}_1^\dagger$, $\hat{\mathbf{e}}_2^\dagger$, and the projectors $\boldsymbol{P}_1 = \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1^\dagger$ and $\boldsymbol{P}_2 = \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2^\dagger$. Verify by direct calculation the *orthonormality* $\hat{\mathbf{e}}_i^\dagger \hat{\mathbf{e}}_j = \delta_{ij}$ and *closure* $\sum_i \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i^\dagger = I$ relations.

d) Use $\vec{x} = I\vec{x} = \boldsymbol{P}_1 \vec{x} + \boldsymbol{P}_2 \vec{x} = \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1^\dagger \vec{x} + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2^\dagger \vec{x} = \hat{\mathbf{e}}_1 x_1 + \hat{\mathbf{e}}_2 x_2$ to find the components x_1 and x_2 of $\vec{x} = (\, 1 1 \,)$ and verify that $\vec{x} = \hat{\mathbf{e}}_1 x_1 + \hat{\mathbf{e}}_2 x_2$.

3. Adjoint Given $\boldsymbol{a} = \begin{pmatrix} 2i \\ 3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ i \end{pmatrix}$, and $\boldsymbol{M} = \begin{pmatrix} 2 & i \\ -i1 & \end{pmatrix}$, show by direct calculation that $\boldsymbol{a}^\dagger \boldsymbol{b} = (\boldsymbol{b}^\dagger \boldsymbol{a})^*$, $(\boldsymbol{M}\boldsymbol{a})^\dagger = \boldsymbol{a}^\dagger \boldsymbol{M}^\dagger$, $(\boldsymbol{a}^\dagger \boldsymbol{M} \boldsymbol{b})^* = \boldsymbol{b}^\dagger \boldsymbol{M}^\dagger \boldsymbol{a}$, and $\boldsymbol{a}^\dagger \boldsymbol{M} = (\boldsymbol{M}^\dagger \boldsymbol{a})^\dagger$. Note the last formula implies that $\boldsymbol{M}$ multiplies to the left in its role as adjoint.