University of Kentucky, Physics 420
Homework #4, Rev. A, due Monday, 2015-10-05

1. The **complex plane** $\{\boldsymbol{z} = (x, y)\}$ is a vector space of *real* and *imaginary* components of $z = x + iy \in \mathbb{C}$ with the additional structure of multiplication, analogous to the *general linear group* $\mathrm{GL}(n)$ of operators. We explore this analogy using a generalization of the exponential map.

a) Which two complex numbers form a basis the the above components of z?

b) Show that the dot and cross product of two points $\boldsymbol{z}_1 = (x_1, y_1)$ and $\boldsymbol{z}_2 = (x_2, y_2)$ are the real and imaginary parts of the complex product $z_1^* z_2 = \boldsymbol{z}_1 \cdot \boldsymbol{z}_2 + i(\boldsymbol{z}_1 \times \boldsymbol{z}_2)_z$, where the $z^* = x - iy$ is the *complex conjugate* of z. Identify the symmetric and antisymmetric parts. Compare the complex $|z|^2 = z^* z = \boldsymbol{z} \cdot \boldsymbol{z}$ and vector $|\vec{\boldsymbol{v}}|^2 = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}$ square magnitudes. Note that the cross product is not closed over the xy-plane: only the z-component is nontrivial!

c) Show graphically that the operator "multiply by i" rotates a point z 90° CCW about the origin.

d) Show graphically that the operator $1 + i\,d\phi : z \mapsto z + iz\,d\phi$ preserves the magnitude of z (assuming $d\phi^2 = 0$), but rotates it CCW by the infinitessimal angle $d\phi$.

e) Obtain a finite rotation by an infinite composition of rotations by $d\phi$, as follows: formally integrate the equation $dz = iz\,d\phi$ with the initial condition $z|_{\phi=0} = z_0$ to obtain the rotation formula $z(\phi) = R_\phi z_0$, with $R_\phi = e^{i\phi}$. Use this property to justify the identity $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n$.

f) Separate the Taylor expansion of $e^{i\phi}$ into $x + iy$ to prove Euler's formula, $e^{i\phi} = \cos\phi + i\sin\phi$.

g) Show that complex multiplication by i is equivalent to the vector operator $\hat{\boldsymbol{z}}\,\times$.

h) Determine the matrix representation M_z of the operator $\hat{\boldsymbol{z}}\,\times$, such that $M_z \boldsymbol{r} = \hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}}$. Do the same for M_x and M_y. Show that $\vec{\boldsymbol{v}}\times = \vec{\boldsymbol{v}} \cdot \boldsymbol{M} = v_x M_x + v_y M_y + v_z M_z$ for any vector $\vec{\boldsymbol{v}}$, and derive the matrix representation of $\vec{\boldsymbol{v}}\,\times$. Note that each of these matrices is antisymmetric, and the stack of three matrices $\boldsymbol{M}$ (cross product tensor) is completely antisymmetric, with components ε_{ijk}.

i) Calculate all nine combinations $M_i M_j$. Show that (M_x, M_y, M_z) have the same properties of Hamilton's quaternions (i, j, k), which extend the rotational properties of complex numbers to 3 dimensions: $i^2 = j^2 = k^2 = ijk = -1$. Do these matrices *commute*?

j) Using the same method as above, show that the matrix for a vector rotation of angle ϕ CCW in the xy-plane can be written $R_\phi = e^{M_z\phi} = I\cos\phi + M_z\sin\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, neglecting trivial z-components. The *exponential* of matrix M_z is defined in term of its Taylor expansion. In general, the matrix for a CCW rotation of angle $v = |\vec{\boldsymbol{v}}|$ about $\hat{\boldsymbol{v}}$ can be written $R_{\boldsymbol{v}} = I\cos v + \boldsymbol{M}\cdot\hat{\boldsymbol{v}}\sin v + \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^T(1 - \cos v)$. The third term handles the non-rotating component parallel to $\hat{\boldsymbol{v}}$.

k) Calculate the eigenvalues and eigenvectors of $M_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to show that $M_z = VWV^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $e^{M_z\phi} = Ve^{W\phi}V^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Multiply this out to verify part j). Thus real Hermitian matrices have real eigenvalues while antiHermitian matrices have Tr=0 and imaginary eigenvalues. The exponential of a Hermitian matrix remains Hermitian, while the exponential of an antiHermitian matrix is an unitary rotation with Det=1 and unit modulus eigenvalues. Amazingly, the Hermitian/antiHermitian decomposition of matrices is analogous to the real/imaginary decomposition of complex numbers.