University of Kentucky, Physics 520
Homework #5, Rev. A, due Monday, 2015-10-12

1. In class we learned that the linear 2nd order **Sturm-Liouville** differential operator

$$L[y(x)] \equiv \frac{1}{w(x)}\left[\frac{d}{dx}p(x)\frac{d}{dx} - q(x)\right]y(x) \tag{1}$$

is self-adjoint, $L^\dagger = L$, with respect to boundary conditions $y(a) = y(b) = 0$ and the inner product

$$\langle y_1|y_2\rangle \equiv \int_a^b w(x)dx\, y_1(x)y_2(x). \tag{2}$$

Thus it has real eigenvalues λ_i and a complete set of orthogonal eigenfunctions $u_i(x)$, so that $L|u_i\rangle = \lambda_i|u_i\rangle$, where $\langle u_i|u_j\rangle = \delta_{ij}$, and $|f\rangle = \sum_i |u_i\rangle f_i$ for any smooth function $f(x) = \langle x|f\rangle$.

a) Show that L is self-adjoint or Hermitian. *Hint:* use the definition $\langle f|H^\dagger|g\rangle \equiv \langle Hf|g\rangle$ to show that the derivative operator $\frac{d}{dx}$ is antiHermitian; apply it to the composition of operators in L.

b) Repeat the proof that $L[u_i] = \lambda_i u_i$, where $\lambda_i \in \mathbb{R}$ and $\langle u_i|u_j\rangle = \delta_{ij}$. Operate L on the general expansion of $|f\rangle$ to show that its *spectral decomposition* is $L = \sum_i \lambda_i|u_i\rangle\langle u_i|$. What is the decomposition of the identity operator $1|f\rangle = |f\rangle$ in the same orthogonal basis $|u_i\rangle$?

c) Legendre polynomials $P_n(\cos\theta)$ are defined by the differential eigenvalue equation $L|f\rangle = \lambda|f\rangle$, where $L[f(\theta)] = \left[\frac{d^2}{d\theta^2} + \cot\theta\frac{d}{d\theta}\right]f(\theta)$. Show that this is a Sturm-Liouville system on the domain $0 < \theta < \pi$, with $w(\theta) = \sin\theta$, $p(\theta) = \sin\theta$, and $q(\theta) = 0$. Change variables to $x = \cos\theta$ and calculate the new functions $w(x)$, $p(x)$, $q(x)$ and domain $a < x < b$. Note the sign change!

d) Show that $\langle x^m|x^n\rangle = \frac{2}{m+n+1}$ if $m+n$ is even and, 0 if $m+n$ is odd. Use the Gram-Schmidt procedure on the basis functions 1, x, x^2, and x^3 to obtain the first four Legendre polynomials $P_\ell(x)$, and find their eigenvalues λ_ℓ.

e) Use your favorite reference to compile a chart of w, p, q, λ_i for the following functions $\phi_i(x)$:

i) azimuthal harmonics $e^{im\phi}$ on $0 < \phi < 2\pi$

ii) associated Legendre functions $P_l^m(x)$ on $-1 < x < 1$

iii) Fourier series $\sin(k_n x)$ on $0 < x < b$

iv) Bessel functions $J_m(x)$ on $0 < x < b$

v) spherical Bessel functions $j_l(x)$ on $0 < x < b$.

2. Simultaneously diagonalize the matrices $A = \begin{pmatrix} -9 & 2 & 6 \\ 2 & -9 & 6 \\ 6 & 6 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 54 & 10 & -3 \\ 10 & -45 & 30 \\ -3 & 30 & 46 \end{pmatrix}$. Are the eigenvectors orthogonal? *Hint:* Octave or Mathematica is your friend!