University of Kentucky, Physics 520
Homework #7, Rev. B, due Monday, 2015-11-09

1. Scattering potential. Let a potential be confined to the region $a < x < b$, ie. $V(x) = 0$ if $x < a$ or $x > b$. Then positive energy scattering states will be plane waves on either side of the potential: $\psi_1(x) = Ae^{ikx} + Be^{-ikx}$ for $x < a$ and $\psi_3(x) = Fe^{ikx} + Ge^{-ikx}$ for $x > b$.

a) Identify the forward and backward travelling plane waves on each side of the potential and calculate the probability current density $j(x) = \frac{i\hbar}{2m}\left[\psi(x)\frac{d}{dx}\psi^*(x) - \psi^*(x)\frac{d}{dx}\psi(x)\right]$ for each.

b) The *scattering matrix* $\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$ is the ratio of incoming and outgoing coefficients. Show that each element forms a reflection or transmission coefficient: $|S_{fi}|^2 = |j_f(x)/j_i(x)|$, where $j_i(x)$ is an incoming current density, and $j_f(x)$ is an outgoing current density.

c) The *transfer matrix* $\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$ is the ratio of left to right coefficients. Calculate the entries of the M in terms of S and vice versa, and check that they are consistent.

d) Calculate the transfer/scattering matrices and forward/backward reflection/transmission coefficients of the following potentials:

i) The δ-potential, $V(x) = -\alpha\delta(x - a)$.

ii) The finite square well, $V(x) = -V_0\theta(a - |x|)$.

iii) The step potential $V(x) = V_0\theta(x)$.

2. A **Double δ-potential** has two identical separated wells $V(x) = -\alpha[\delta(x + a) + \delta(x - a)]$.

a) Calculate the bound state wave functions of this potential.

b) Show that the transfer matrix for two potential wells is the product of transfer matrices for each well $M = M_1 M_2$. Multiply the transfer matrices of the two single wells to calculate the transmission and reflection coefficients probabilities of $V(x)$.

3. [**bonus**] Classical wave media are characterized by **velocity** v and **characteristic impedance** Z_0. For waves on a string of tension T and density μ, $v = \sqrt{T/\mu}$ and $Z_0 = \sqrt{T\mu}$. For EM waves, $v = 1/\sqrt{\mu\epsilon}$ and $Z_0 = \sqrt{\mu/\epsilon}$. A transmission line is characterized by the propagation constant $\gamma = \sqrt{ZY}$ so that $V(z) = V_0 e^{-\gamma z - i\omega t}$, and $Z_0 = \sqrt{Z/Y}$, where $Z = R + i\omega L$ and $Y = G + i\omega C$ are the series impedance and shunt admittance of the line, respectively.

a) Compare the velocity and reflection/transmission coefficients of a QM plane waves on a step potential $V(x) = V_0\theta(x)$ to obtain the "propagation constant" and "characteristic impedance".

b) Compare a point mass on a string to the δ-potential to obtain the QM point impedance.

c) The physical interpretation of impedance is the ratio the two fields involved in the boundary conditions: $f(x)$ and $f'(x)$ for a string; **E** and **H** for an EM wave; V and I for a wave guide. Note that the product of these two fields is the transfer of energy $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. Is there a corresponding interpretation for quantum mechanical waves?