L07-Wavepackets: complementarity

Friday, September 11, 2015 9:57 AM

* review plane waves (L06-de Broglie waves: Interference)

- they are the building blocks of wave packets
- put together by superposition → quantum interference

* review PhET applet (Fourier analysis)

- $k_1 = \frac{2\pi}{\lambda_1}$ fundamental frequency period λ_1
  frequency of the wave packets [discrete only]
  $k_1 = dk \to 0$ for Fourier transform
- A_n peaked at $n = k_0 / k_1$ carrier (phase) frequency
  apparent frequency of the wave
- Δk = frequency bandwidth (modulation)
  sharpness of packet in position space
  highest frequency of wave packet

* development of Fourier Series & Transform

- follow Binney & Skinner, App. C.
- orthonormality easy to prove closure difficult

- Suppose $f(\phi) = \sum_{n=-\infty}^{\infty} a_n e^{in\phi}$

dummy index

then $\langle n|f\rangle \equiv \int_{-\pi}^{\pi} d\phi\, e^{-in\phi} f(\phi) = \int e^{-in\phi} \left(\sum_{n'} a_{n'} e^{in'\phi} \right)$

normalization

$$= \sum_n a_n \int d\phi\, e^{i(n'-n)\phi} = \sum_n a_{n'}\, 2\pi\, \delta_{nn'} = 2\pi\, a_n$$

- using orthogonality: $\boxed{\langle n|n'\rangle = 2\pi\,\delta_{nn'}}$

$$\langle n|n'\rangle \equiv \int d\phi\, e^{i(n'-n)\phi} = \frac{1}{i(n-n')} e^{i(n'-n)\phi}\Big|_{-\pi}^{\pi} = 0 \text{ if } n \neq n'$$

$$\langle n|n\rangle = \int d\phi\, e^{0} = 2\pi$$ normalization

$\Rightarrow 2\pi\,\delta_{nn'}$

- how do we know we can expand $f(\theta) = \sum a_n e^{in\phi}$

$$f(\phi) = \sum_n a_n e^{in\phi} = \sum_n \left[\frac{1}{2\pi}\int d\phi' e^{-in\phi'} f(\phi')\right] e^{in\phi}$$

$$= \int d\phi' f(\phi') \underbrace{\sum_n \frac{1}{2\pi} e^{in(\phi-\phi')}}_{\delta(\phi-\phi')} = f(\phi)$$

- closure $\boxed{\sum_n |n\rangle\langle n| = I}$

for then: $f(\phi) = \int d\phi' f(\phi') \delta(\phi-\phi') = \sum_n a_n e^{in\phi}$

- notes: a) 2π normalization appears every where
b) can extend by change of variables and limits.

- interval $0 < x < L$: $\phi = 2\pi x/L$ $|n\rangle \sim e^{2\pi i n x/L}$

$$\langle n|n'\rangle = \int_{-L/2}^{L/2} dx\, e^{2\pi i(n-n')x/L} = L\,\delta_{nn'}$$

$$\sum |n\rangle\langle n'| \sim \sum_n \frac{1}{L} e^{2\pi i n(x-x')/L} = \delta(x-x')$$

- harmonic # $n \rightarrow$ freq. $k_n = \frac{2\pi n}{L} \equiv \underbrace{dk}_{k_1 = 2\pi/L} \cdot n$

$$\langle k_n|k_{n'}\rangle = \int_{-L/2}^{L/2} dx\, e^{i(k_n-k_{n'})x} = L\,\delta_{nn'}$$

$$\sum |k_n\rangle\langle k_n'| \sim \sum_n \frac{dk}{2\pi} e^{ik_n(x-x')} = \delta(x-x')$$

- limit as $L \rightarrow \infty$ $dk \rightarrow 0$ $k_n \rightarrow k$ $\sum_n dk_n \rightarrow \int dk$

$$\langle k|k'\rangle \equiv \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\,\delta(k-k')$$

$$\int \frac{dk}{2\pi} |k\rangle\langle k| \sim \int \frac{dk}{2\pi} e^{ik(x-x')} = \delta(x-x')$$

- stick one of these in to obtain a transformation

* useful integrals: $I_n = \int_0^\infty dx\, x^n e^{-\alpha x^2}$

$$I_1 = \int_0^\infty x e^{-\alpha x^2} dx = \frac{1}{2\alpha}\int_0^\infty e^{-u}\, du = \frac{1}{2\alpha} \qquad \begin{array}{l} u = \alpha x^2 \\ du = \alpha \cdot 2x dx \end{array}$$

$$I_0^{\,2} = \int_0^\infty\!\!\int_0^\infty dx\,dy\; e^{-\alpha(x^2+y^2)} = \int_0^\infty \frac{\pi}{2} s\,ds\; e^{-\alpha s^2} = \frac{\pi}{2} I_1 = \frac{\pi}{4\alpha}$$

$$I_{n+2} = \int_0^\infty x^{n+2} dx\, e^{-\alpha x^2} = -\frac{\partial}{\partial\alpha} I_n \qquad I_2 = -\frac{\partial}{\partial\alpha}\sqrt{\frac{\pi}{4\alpha}} = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

n	0	1	2	3	4	5	...
$2I_n$	$\sqrt{\frac{\pi}{\alpha}}$	$\frac{1}{\alpha}$	$\frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$	$\frac{1}{\alpha^2}$	$\frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$	$\frac{2}{\alpha^3}$	...

$$\Gamma(t) \equiv \int_0^\infty u^{t-1} e^{-u}\, du \qquad \text{let } u = \alpha x^2, \quad du = \alpha \cdot 2x dx$$

$$= \int_0^\infty (\alpha x^2)^{t-1} e^{-\alpha x^2} \cdot \alpha \cdot 2x\, dx$$

$$= 2\alpha^t \int_0^\infty x^{2t-1} e^{-\alpha x^2} dx \qquad \Gamma(t+1) = t\,\Gamma(t)$$

$$\Gamma(n+1) = n!$$

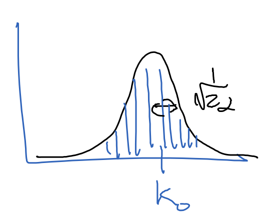
$$= 2\alpha^t I_{2t-1} \qquad \begin{array}{l} n = 2t-1 \\ t = \frac{1}{2}(n+1) \end{array} \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

so $$2I_n = \frac{\Gamma(\frac{1}{2}(n+1))}{\alpha^{\frac{1}{2}(n+1)}}$$

* Uncertainty - following Gasiorowicz 2-6

$$A(k) = e^{-\alpha(k-k_0)^2/2} \qquad \alpha = \frac{1}{2\sigma_k^2} \qquad \Delta k = \frac{1}{\sqrt{2\alpha}}$$

$$\psi(x) = \int dk\, A(k)\, e^{ikx}$$



$$= \int dk\; e^{-\alpha(k-k_0)^2/2 + ikx} \qquad q = k - k_0$$

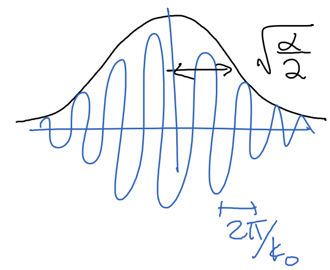
$$= e^{ik_0x} \cdot \int dq\; e^{-\alpha q^2/2 + iqx} \qquad -\alpha/2\left(q - \frac{i}{\alpha}x\right)^2 - \frac{x^2}{2\alpha}$$

$$= e^{ik_0x}\cdot\int dq\; e^{-\alpha q^2/2 + iqx} \qquad -\frac{\alpha}{2}\big(\underbrace{q - \tfrac{i}{\alpha}x}\big)^2 - \frac{x^2}{2\alpha}$$

$$= e^{ik_0x - \frac{x^2}{2\alpha}} \int dq'\; e^{-\alpha/2\, q'^2} \qquad q' = q - \frac{i}{\alpha}x$$

$$= \underbrace{e^{ik_0x}}_{\text{phase}}\;\underbrace{e^{-\frac{x^2}{2\alpha}}}_{\text{ampl.}}\cdot\underbrace{\sqrt{\frac{2\pi}{\alpha}}}_{\text{norm}} \qquad \Delta x = \sqrt{\frac{\alpha}{2}}$$

$$\boxed{\Delta x\,\Delta k = \frac{1}{2}}$$



$$\Big[\underbrace{\int dx\, e^{-\alpha x^2}}_{I}\Big]^2 = \int_0^\infty 2\pi s\,ds\, e^{-\alpha s^2} = \pi/\alpha \int_0^\infty du\, e^{-u} = \pi/\alpha$$

$$dx\,dy = 2\pi s\,ds \qquad u = \alpha s^2 \quad du = 2\alpha s\,ds \qquad \boxed{\int dx\, e^{-\alpha x^2} = \sqrt{\pi/\alpha}}$$