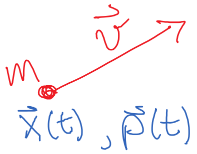
L08-Schrödinger Equation: Dispersion

Monday, September 14, 2015 9:11 AM

* continuing saga of particle-wave duality.
  - talked about photons $E=h\nu$ & deBroglie $p=h/\lambda$
  - talked about wave packets & uncertainty
  - today: dispersion (propagation of waves)
  - next: interaction (observation, probability amplitude)

* quantization / dispersion chart
  - propagation of particles: $m, \vec{v}, \vec{p}, E$ (DOF: $\vec{x}, \vec{p}$)
  - propagation of plane waves: $v_\phi(\vec{k},\hat{n})$ $\vec{k}, \omega$ (DOF: $A_{\vec{k}}$)
  - compare dispersion relations

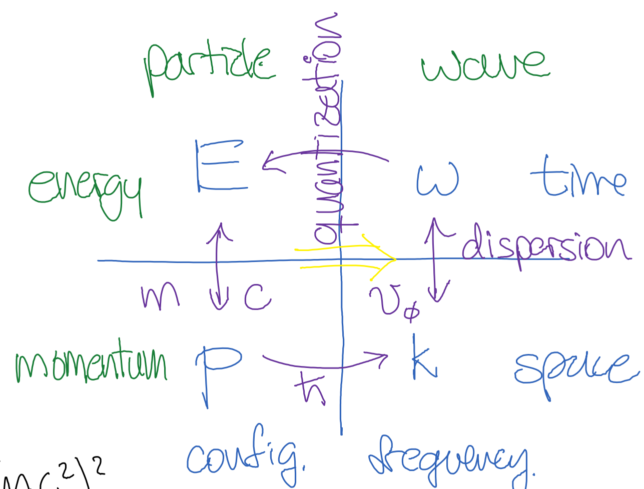


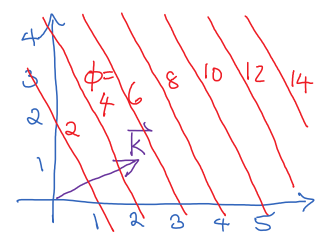
$E = \frac{1}{2}mv^2 \rightarrow \gamma mc^2$

$\vec{p} = m\vec{v} \rightarrow \gamma m v$

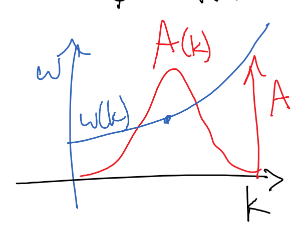
$E = \frac{p^2}{2m}$ ↘

$E^2 - (pc)^2 = (mc^2)^2$





$v_\phi = \frac{\omega}{\vec{k}\cdot\hat{n}} = c$



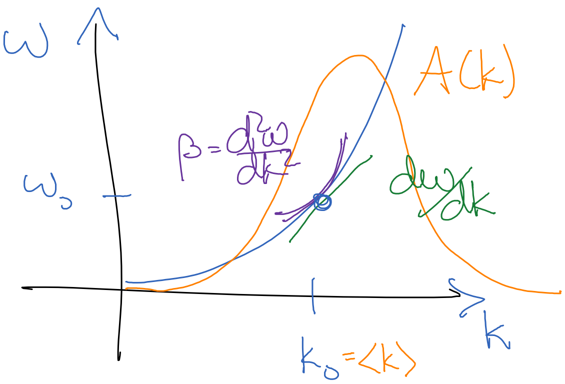
- different dispersion relations:
  - why not $c = \frac{\omega}{k} = \frac{E}{p}$ for a particle of mass m?
  - what velocity to use?

$$E = \hbar\omega = \frac{(\hbar k)^2}{2m} = p^2/2m \qquad v_\phi = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{1}{2}v_{particle} \;?$$

* dispersion

* dispersion

$$\omega(k) = \omega_0 + \left.\frac{d\omega}{dk}\right|_{k_0}(k-k_0) + \frac{1}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k_0}(k-k_0)^2$$



$$kx - \omega t = k_0 x + (k-k_0)x$$

$$+ \underbrace{\omega_0 t}_{\omega(k_0)} + \left.\frac{d\omega}{dk}\right|_{k_0}(k-k_0)t + \frac{1}{2}\left.\frac{d\omega^2}{dk^2}\right|_{k_0}(k-k_0)^2 t$$

$$= (k_0 x - \omega_0 t) + (k-k_0)\left(x - \left.\frac{d\omega}{dk}\right|_{k_0} t\right) + \frac{1}{2}(k-k_0)^2 \left.\frac{d^2\omega}{dk^2}\right|_{k_0} t$$

$$\psi(x,t) = \int dk\, A(k)\, e^{i\,kx - \omega(k)t}$$

$$= e^{i\underbrace{(k_0 x - \omega_0 t)}_{\phi_0}} \cdot \int dq\, A(q+k_0)\, e^{iq\underbrace{(x - v_g t)}_{x_t} + \frac{1}{2}q^2 \underbrace{\left.\frac{d^2\omega}{dk^2}\right|_{k_0}}_{-\beta} t}$$

$$= \underbrace{e^{i\phi_0}}_{\substack{\text{phase velocity} \\ v_\phi = \frac{\omega_0}{k_0}}} \int dq\, A(q,t) \underbrace{e^{iqx_t}}_{\substack{\text{group velocity} \\ v_g = \nabla_k \omega_0}} \qquad A(q,t) = A(q+k_0)\, \underbrace{e^{-\frac{1}{2}q^2 \beta t}}_{\text{dispersion spreading out}}$$

thus $v_\phi = \left.\frac{\omega}{k}\right|_{k_0}$, $v_g = \left.\frac{d\omega}{dk}\right|_{k_0}$, $\beta = \left.\frac{d^2\omega}{dk^2}\right|_{k_0}$

phase, group velocity, dispersion

* simple example: beats

$$A_1 e^{i(k_1 x - \omega_1 t)} + A_2 e^{i(k_2 x - \omega_2 t)} = \ldots$$

let $\bar{k} = \frac{1}{2}(k_1 + k_2)$ $\quad \bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$

$\hat{k} = \frac{1}{2}(k_1 - k_2)$ $\quad \hat{\omega} = \frac{1}{2}(\omega_1 - \omega_2)$

* momentum of a plane wave — "operate to observe"

$$\boxed{\frac{d}{dx}}\, e^{ikx} = ik\, e^{ikx} \qquad \boxed{\frac{d}{dt}}\, e^{-i\omega t} = -i\omega\, e^{-i\omega t}$$

operator eigenvalue operator eigenvalue

- wave equation: $\omega^2 = k^2 c^2$ $\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\psi = 0$
- Schrödinger: $E = \frac{p^2}{2m} + V$ $\boxed{-i\hbar\partial_t \Psi = \frac{-\hbar^2}{2m}\nabla^2\Psi + V\,\Psi}$
  - spatial & temporal frequencies correspond to local momentum & energy at each point.