# L09-Born Probability, Ehrenfest theorem

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* probabilistic interpretation of wave function:
  $\psi(x)$ = probability amplitude (quantum interference)
  $|\psi(x)|^2$ = probability density (classical behavior)
  $\psi^*(x)\psi(x)\,dx = |\psi(x)|^2 dx$ probability of being observed
  - Imediately after observation, the wave function collapses to sharply peaked around $x_0$: $\psi(x) = \delta(x-x_0)$ so that a second measurement will observe the same result.

* review quantization/dispersion from L08
  — essentially the Schrödinger Eq!

* Conservation of probability: (follow Griffiths 1.4) (Gasiorowicz 2-6)

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\,|\psi|^2 = \int_{-\infty}^{\infty} dx\,\frac{\partial}{\partial t}|\psi(x,t)|^2$$

convert $\partial_t$ to $\partial_x$:

$$\partial_t\left(|\psi|^2 = \psi^*\psi\right) = \psi^*\cdot\partial_t\psi + \partial_t\psi^*\cdot\psi$$

$$\partial_t\psi = \frac{i\hbar}{2m}\partial_x^2\psi - \frac{i}{\hbar}V\psi$$

[schrödinger]

$$\partial_t\psi^* = \frac{-i\hbar}{2m}\partial_x^2\psi^* + \frac{i}{\hbar}V^*\psi^*$$

$*$: $i \to -i$
$\psi = \psi_{real} + i\,\psi_{im}$
$\psi^* = \psi_{real} - i\,\psi_{im}$
$V^* = V$ (real)

$$\partial_t|\psi|^2 = \frac{i\hbar}{2m}\left(\psi^*\partial_x^2\psi - \partial_x^2\psi^*\cdot\psi\right)$$

$$= \frac{\partial}{\partial x}\underbrace{\frac{i\hbar}{2m}\left(\psi^*\cdot\partial_x\psi - \partial_x\psi^*\cdot\psi\right)}_{j(x)}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\,|\psi|^2 = \int_{-\infty}^{\infty} dx\,\frac{\partial}{\partial x}j(x) = j(x)\Big|_{-\infty}^{\infty} \to 0$$

$\psi(\pm\infty) \to 0$

- continuity eq: $\partial_t \rho + \nabla\cdot\vec{j} = 0$ (conservation of *)

$$\partial_t \underbrace{(\psi^*\psi)}_{\rho(x)} + \partial_x \underbrace{\left(\psi^* \frac{\hat{p}}{2m}\psi - \psi\frac{\hat{p}}{2m}\psi^*\right)}_{\vec{j}(x)} = 0$$

no time derivatives!

- similar to Poynting theorem in E&M:

dynamical equations

$$\partial_t u = \partial_t\left(\tfrac{1}{2}\vec{D}\cdot\vec{E} + \tfrac{1}{2}\vec{B}\cdot\vec{H}\right) = \frac{\partial\vec{D}}{\partial t}\cdot\vec{E} + \frac{\partial\vec{B}}{\partial t}\cdot\vec{H} = (\nabla\times\vec{H} - \vec{J})\cdot\vec{E} + (-\nabla\times\vec{E})\cdot\vec{H}$$

$$= -J\cdot E - \nabla\cdot(\vec{E}\times\vec{H}) = -\frac{dw}{dt} - \nabla\cdot\vec{S}$$

← conserved current.

* probability density, current for plane wave:

$$\psi = Ae^{\overbrace{i(kx-\omega t)}^{\phi}} \qquad \psi^* = A^* e^{-\overbrace{i(kx-\omega t)}^{\phi}} \qquad \partial_x\psi = ik\psi$$

$$\rho(x) = |\psi|^2 = \psi^*\psi = A^*A\, e^{i(\cancel{\phi}-\cancel{\phi})} = |A|^2 = \text{const!}$$

$$j(x) = \psi^*\frac{-i\hbar}{2m}\partial_x\psi - \psi\frac{-i\hbar}{2m}\partial_x\psi^*$$

$$= \psi^*\frac{\hbar k}{2m}\psi - \psi\frac{-\hbar k}{2m}\psi^*$$

$$= \frac{\hbar k}{m}|A|^2 = \vec{v}\,|A| = \vec{v}\,\rho(x)$$

* Ehrenfest theorems for $x, p$

1) $\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m}$ (Griffiths 1.5 / Gasiorowicz 2-7)

$$\frac{d}{dt}x = \frac{\partial}{\partial p}H$$

$$p\,dx = H\,dt$$

$$= \int_{-\infty}^{\infty} dx\; \partial_t\psi^*\cdot x\psi + \psi^* x\,\partial_t\psi$$

$$= \int dx\, \underbrace{\left[\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \cancel{\frac{V}{-i\hbar}\psi^*}\right]x\psi}\, + \psi^* x\left[\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi + \cancel{\frac{V}{i\hbar}\psi}\right]$$

[Schrödinger]

$$\frac{\partial^2\psi^*}{\partial x^2}x\psi = \frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}x\psi\right) - \frac{\partial\psi^*}{\partial x}\underbrace{\frac{\partial}{\partial x}(x\psi)}$$

[integration by parts]

$$= \frac{\partial}{\partial x}\left(\frac{\partial \psi^*}{\partial x} x\psi\right) - \frac{\partial}{\partial x}(\psi^*\psi) + \psi^*\frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x}(\psi^* x \psi) + \psi^* \frac{\partial}{\partial x}\Big(\overbrace{x\frac{\partial \psi}{\partial x}}^{\psi + x\frac{\partial\psi}{\partial x}}\Big)$$

$$= \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \psi^*}{\partial x} x\psi - \psi^*\psi - \psi^* x \psi\right)}_{\int_{-\infty}^{\infty} d(\dots) = 0} + 2\psi^*\partial_x\psi + \underbrace{\psi^* x \frac{\partial^2\psi}{\partial x^2}}_{\text{cancels 2nd term above}} \qquad \frac{\partial\psi}{\partial x} + x\frac{\partial^2\psi}{\partial x^2}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, \psi^* \underbrace{x}_{\hat{x}} \psi = \frac{1}{m}\int_{-\infty}^{\infty} \psi^* \underbrace{[-i\hbar\partial_x]}_{\hat{p}} \psi$$

$$\frac{d}{dt}\langle x\rangle = \frac{1}{m}\langle p\rangle$$

Expectation value using operators.

2) $\frac{d}{dt}\langle p\rangle = \langle F\rangle = \langle -\frac{\partial V}{\partial x}\rangle$

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, \psi^*(-i\hbar\partial_x)\psi = \int_{-\infty}^{\infty} dx\, (-i\hbar)\left[\partial_t\psi^* \cdot \partial_x\psi + \psi^*\partial_x\partial_t\psi\right]$$

$$= \int_{-\infty}^{\infty} dx \left[\underbrace{\frac{-\hbar^2}{2m}\partial_x^2}_{T}\psi^* + V\psi^*\right]\partial_x\psi + \psi^*\partial_x\left[\underbrace{\frac{\hbar^2}{2m}\partial_x^2}_{-T}\psi - V\psi\right]$$

$$= \int_{-\infty}^{\infty} dx\, \cancel{(\psi\hat{T}\psi^* - \psi^*\hat{T}\psi)} + \cancel{V\psi^*\partial_x\psi} - \psi^*\left[\partial_x V\cdot\psi + \cancel{V\partial_x\psi}\right]$$

see below

$$= \int_{-\infty}^{\infty} dx\, \psi^* - \frac{\partial V}{\partial x}\psi = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

$*$ Real expectation values of $p$, $p^2$, $T = p^2/2m$

$$\langle p\rangle = \int_{-\infty}^{\infty} dx\, \psi^*\hat{p}\psi = \int_{-\infty}^{\infty} dx\, \psi^*(-i\hbar\partial_x)\psi = -i\hbar\int_{-\infty}^{\infty}\psi^* d\psi$$

$$= -i\hbar\, \cancel{\psi^*\psi\Big|_{-\infty}^{\infty}}^{\,0} + i\hbar\int_{-\infty}^{\infty}\psi\, d\psi^*$$

$$= \int_{-\infty}^{\infty} dx\, \psi(-i\hbar\partial_x\psi)^* = \int dx\, (\hat{p}\psi)^*\psi = \langle p\rangle^*$$

$$\langle p^2\rangle = \int dx\, \psi^* p^2\psi = \int dx\, (\hat{p}\psi)^* p\psi = \int dx\, (\hat{p}^2\psi)^*\psi = \langle p^2\rangle^*$$

$$\langle T \rangle = \int dx\, \psi^* \frac{p^2}{2m} \psi = \int dx\, \psi^* \frac{-\hbar^2}{2m} \partial_x^2 \psi = \langle T \rangle^*$$