L11-Operators: Rotations

Monday, September 21, 2015 6:35 AM

* review basis: orthonormality & closure:

$1\cdot 1 = 1$! identity transform: $\frac{2.54\,\text{in}}{1\,\text{cm}} = 1$, also $\vec{\mathbb{B}}$, $\mathbb{G}$, dx, $\vec{\sigma}$!

$$\boxed{\begin{aligned}\vec{x} &= \vec{\mathbb{B}}\,\mathbb{x} \\ \vec{x} &= \sum_i \hat{e}_i x_i \\ |f\rangle &= \int dx\, |x\rangle f(x)\end{aligned}}$$

vectors = basis · components

$\vec{x}\cdot\vec{y} = \mathbb{x}^T(\vec{\mathbb{B}}^T\cdot\vec{\mathbb{B}})\,\mathbb{y}$

metric

$\mathbb{G}\to\mathbb{I}$

$$\boxed{\begin{aligned}\vec{\mathbb{B}}\cdot\vec{\mathbb{B}} &= \mathbb{I} \\ \hat{e}_i\cdot\hat{e}_j &= \delta_{ij} \\ \langle x|x'\rangle &= \delta(x-x')\end{aligned}}$$

orthonormality

$$\boxed{\begin{aligned}\vec{\mathbb{B}}\,\vec{\mathbb{B}}\cdot &= 1 \\ \sum \hat{e}_i\hat{e}_i\cdot &= 1 \\ \int dx\,|x\rangle\langle x| &= 1\end{aligned}}$$

closure

– how do we use them? vectors $\longleftrightarrow$ components

$\vec{x} = \vec{\mathbb{B}}\,\vec{\mathbb{B}}\cdot\vec{x} = \vec{\mathbb{B}}\,\mathbb{x}$ $\qquad$ $\mathbb{x} = \vec{\mathbb{B}}\cdot\vec{\mathbb{B}}\,\mathbb{x} = \tilde{\mathbb{B}}\,\vec{x}$

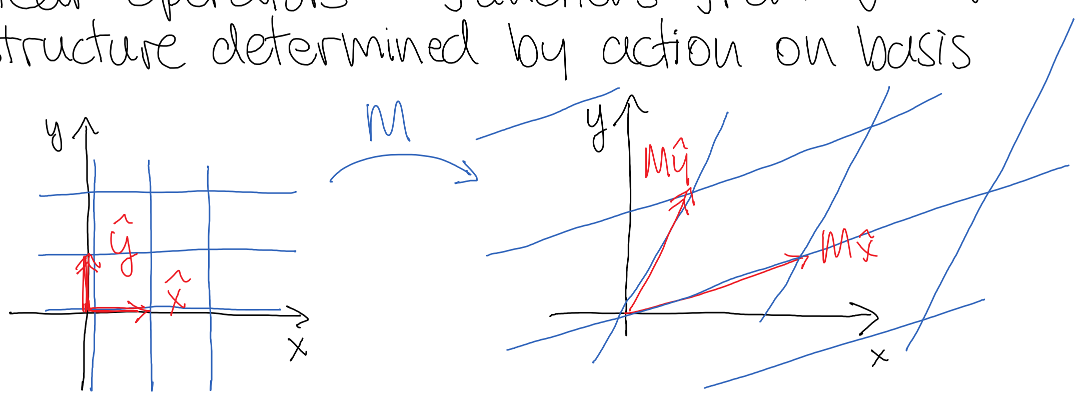
$\tilde{\mathbb{B}} : \vec{x} \mapsto \mathbb{x}$ $\qquad$ $\vec{\mathbb{B}} : \mathbb{x} \to \vec{x}$

$\underbrace{\vec{\mathbb{B}}\,\vec{\mathbb{B}}\cdot}_{\vec{\mathbb{I}}} : \vec{x} \mapsto \mathbb{x} \mapsto \vec{x}$ $\qquad$ $\underbrace{\tilde{\mathbb{B}}\,\vec{\mathbb{B}}}_{\mathbb{I}} : \mathbb{x} \mapsto \vec{x} \mapsto \mathbb{x}$

$\vec{\mathbb{I}} : V \to V$ $\qquad$ $\mathbb{I} : \mathbb{R}^n \to \mathbb{R}^n$

* linear operators – functions from $V \to V$

– structure determined by action on basis



* components

$\vec{w} = M(\vec{v}) = M(\vec{\mathbb{B}}\,\mathbb{v})$

$\vec{\mathbb{B}}\,\mathbb{w} = M(\vec{\mathbb{B}})\,\mathbb{v} = \vec{\mathbb{B}}\,\mathbb{M}\,\mathbb{v}$

$\mathbb{w} = \mathbb{M}\,\mathbb{v}$ $\quad$ where $\boxed{\mathbb{M} = \vec{\mathbb{B}}\cdot M(\vec{\mathbb{B}})}$

$$\underbrace{\tilde{\mathbb{B}}(\vec{w})}_{\mathbb{w}} = \underbrace{\tilde{\mathbb{B}}(M(\vec{\mathbb{B}}))}_{\mathbb{M}}\overbrace{\;}^{\text{closure}}\underbrace{\tilde{\mathbb{B}}(\vec{v})}_{\mathbb{v}}$$

$\mathbb{M}_{ij} = \hat{e}_i\cdot M(\hat{e}_j) \sim \langle x|A|y\rangle$

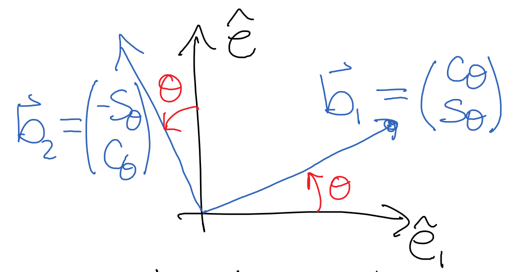
(matrix elements)

matrix operators inherit 2 bases:

from domain & range!

from domain & range!

* Rotations (active)



let $\vec{b}_1 = R\,\hat{e}_1 \quad \vec{b}_2 = R\,\hat{e}_2$

combine: $(\vec{b}_1\,\vec{b}_2) = R(\hat{e}_1\,\hat{e}_2)$

$\vec{\mathbb{b}} = R\,\hat{\mathbb{e}}$

$$\begin{pmatrix} b_1^x & b_2^x \\ b_1^y & b_2^y \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- rotations are "orthogonal" (preserve the metric)

$$\vec{\mathbb{b}}^T \cdot \vec{\mathbb{b}} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix} \cdot (\vec{b}_1\,\vec{b}_2) = \mathbb{I} \qquad R^T R = \mathbb{I}$$

$\vec{b}_i \cdot \vec{b}_j = \delta_{ij}$ in general, $R^{adj} \cdot R = \mathbb{I}$, or $\boxed{R^\dagger G R = G}$

* Hermitian adjoint of a operator (complex transpose)
  - how an operator commutes with the metric

$$M(\vec{a}) \cdot \vec{b} \equiv \vec{a} \cdot M^{adj}(\vec{b}) \qquad \langle Mf | g \rangle = \langle f | M^\dagger g \rangle$$

$$(M\,a)^\dagger G\,b = a^\dagger G\,M^{adj}\,b$$

$$M^\dagger G = G\,M^{adj} \qquad \boxed{M^{adj} = G^{-1} M^\dagger G} \rightarrow M^\dagger \text{ if } G = \mathbb{I}$$

- a matrix multiplies "to the left" as its adjoint

if $\mathbb{b} = \mathbb{a} M$ $\quad (b_1\,b_2) = (a_1\,a_2)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}^* = \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}^* \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^*$

then $\mathbb{b}^\dagger = M^\dagger \mathbb{a}^\dagger$

note the transpose

- Adjoint is a duality: $(M^{adj})^{adj} = M$

- Adjoint of product: $(A \cdot B)^{adj} = B^{adj} \cdot A^{adj}$ like $(AB)^{-1} = B^{-1}A^{-1}$

- Unified operation ($\dagger$) for adjoint of operators, vectors, scalars!

* Active vs. Passive rotations
  - Active rotations: physically move vectors

- only one basis, $\vec{b}_i$ are just rotated vectors.

$\vec{v}' = R\vec{v}$ $\quad\quad \vec{v} = \hat{x}v_x + \hat{y}v_y \rightarrow \vec{v}' = \hat{x}v_x' + \hat{y}v_y'$

- example: evolution of the wavefunction in time

– Passive rotations (identity transform) $\vec{v}$ stays put
- turn your head and look at it from different angle.
- new components correspond to a change of basis

$$\vec{V} = \underbrace{\overbrace{\vec{\mathbb{b}}}\,\mathbb{v}'}\ = \overbrace{\hat{\mathbb{e}}\,\underbrace{\mathbb{R}\,\mathbb{v}}} = \hat{\mathbb{e}}\,\mathbb{v} = \vec{V} \text{ (still)}$$

$$\vec{\mathbb{b}} = \hat{\mathbb{e}}\mathbb{R} \quad \Rightarrow \quad \mathbb{v}' = \mathbb{R}\,\mathbb{v}$$

$$\underbrace{\vec{V} = \mathbb{1}\,\vec{V}}_{\text{identity transform}} \qquad \underbrace{\vec{\mathbb{b}}^T\cdot\vec{V}}_{\mathbb{v}'} = \underbrace{\vec{\mathbb{b}}^T\cdot\mathbb{1}\,\hat{\mathbb{e}}}_{\mathbb{R}}\overbrace{}^{\text{closure}}\underbrace{\hat{\mathbb{e}}^T\cdot\vec{V}}_{\mathbb{v}}$$

(wait for L13)

- example: Fourier transform: $\langle x|f\rangle = \int dk \langle x|k\rangle\langle k|f\rangle$

– Similarity transformation: change of basis of a matrix

if $\mathbb{M} = \hat{\mathbb{e}}^T\cdot M(\hat{\mathbb{e}})$ are matrix elements in the basis $\hat{\mathbb{e}}$,

then $\mathbb{M}' = \vec{\mathbb{b}}^{-1}\cdot M(\vec{\mathbb{b}}) = \underbrace{\vec{\mathbb{b}}^{-1}\cdot\hat{\mathbb{e}}}_{\mathbb{B}^{-1}}\underbrace{\hat{\mathbb{e}}^T\cdot M(\hat{\mathbb{e}}}_{\mathbb{M}}\underbrace{\hat{\mathbb{e}}^T\cdot\vec{\mathbb{b}}}_{\mathbb{B}})$ in the basis $\vec{\mathbb{b}}$

thus $\boxed{\mathbb{M}' = \mathbb{B}^{-1}\mathbb{M}\mathbb{B}}$ change-of-basis formula for matrices

to transform the domain, $\mathbb{B}^{-1}$ the range.

$\mathbb{B} = \hat{\mathbb{e}}^T\cdot\mathbb{1}\,\vec{\mathbb{b}}$ transforms the domain ($\vec{v}$)

$\mathbb{B}^{-1} = \vec{\mathbb{b}}^{-1}\cdot\mathbb{1}\,\hat{\mathbb{e}}$ transforms the range ($\vec{w}$)