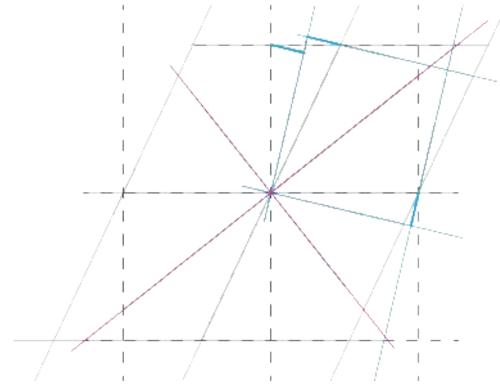
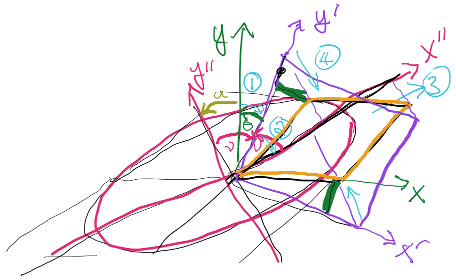
Characterization of Linear Operators

Sunday, October 4, 2015 8:44 AM



* Polar decomposition

- any matrix M can be expressed as a rotation R and stretch S



Rotation to sym. R ① $R=\begin{pmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{pmatrix}$

Eigenvec to ellipse V ② $V=\begin{pmatrix} C_\nu & -S_\nu \\ S_\nu & C_\nu \end{pmatrix}$

Eigenval W ③④ $W=\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$

$\left.\begin{matrix} V \\ W \end{matrix}\right\} S$

$S = VWV^{-1} = \begin{pmatrix} x & a \\ a & y \end{pmatrix}$

$M = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

- example: sheer transformation (see ppt)
- the stretch $S = VWV^\dagger$ is diagonal after further rotation
- combine rotations: singular value decomposition (SVD)

$$M = RS = RVWV^\dagger = UWV^\dagger \quad \text{where} \quad U^\dagger U = I \quad V^\dagger V = I$$

* Diagonalization - notation of eigensystems

$$M\vec{u}_i = \lambda_i \vec{u}_i \qquad U = \begin{pmatrix} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

$$MU = UD$$

$$U^\dagger M U = D \qquad U^\dagger U = I, \quad \text{otherwise } U^{-1} M U = D$$

similarity transform - change of basis (rotation)

$$M = UDU^\dagger = \sum_i \lambda_i \overbrace{\vec{u}_i \vec{u}_i^\dagger}^{\text{projector } P_i} \qquad I = UU^\dagger = (\vec{u}_1\, \vec{u}_2 \cdots)\begin{pmatrix} \vec{u}_1^\dagger \\ \vec{u}_2^\dagger \\ \vdots \end{pmatrix} = \sum_i \overbrace{\vec{u}_i \vec{u}_i^\dagger}^{\text{projector } P_i}$$

* Exponential - Normal Matrix analogy.

- a square matrix G can be decomposed into symmetric T and antisymmetric A parts

$$\tfrac{1}{2}(M + M^\dagger) \equiv T$$
$$\tfrac{1}{2}(M - M^\dagger) \equiv A$$
$$M = T + A$$

with respect to the adjoint

- take the exponential of each: $T^\dagger = T$ Hermitian $A^\dagger = -A$ anti Hermitical

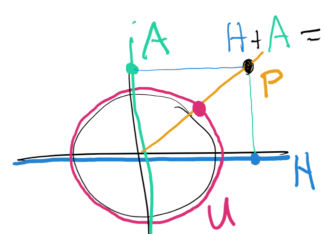
a) $T = VDV^\dagger$ because $T^\dagger = T$
$S \equiv e^T = V e^D V^\dagger$ semi positive definite: (+) eigenvalues

b) $R \equiv e^A$ $R^\dagger R = e^{-A} \cdot e^A = e^0 = I$ unitary
$\mathrm{Det}\, R = \mathrm{Det}\, e^A = e^{\mathrm{Tr}\, A} = |e^{i\theta}| = 1$

c) $M = e^G$ ← generator if $[T, A] = 0$ (M is normal), then
$= e^{T+S} = e^T \cdot e^S = S \cdot R$ polar decomposition

- summary: complex analogy

$\exp(G = T + A)$ $\exp(\omega = \tau + i\phi)$ $\alpha = i\phi$
$= (M = S \cdot R)$ $= (z = s \cdot r)$ $s = e^\tau$ $r = e^{i\phi}$



a) Normal $N = H + iK$ $H^\dagger = H$ $K^\dagger = K$
$NN^\dagger = N^\dagger N$ $HK = KH$ → n complex eigenvalues.

b) (anti) Hermitian $H^\dagger = H$ $A^\dagger = (iK)^\dagger = -iK = -A$
→ (imaginary) real eigenvalues

c) Positive definite: positive eigenvalues

c) Unitary: $U^\dagger U = I$ unit eigenvalues. $e^{i\theta_i}$

- Proofs: if $f(x) = \sum_i a_i x^i$ and $M = UDU^{-1}$ then

note $\underbrace{UDU^{-1}U}_{I}DU^{-1} = UD^2U^{-1}$

$$f(M) \equiv \sum_i a_i M^i = \sum_i a_i (UDU^\dagger)^i$$
$$= U\left(\sum_i a_i D^i\right)U^\dagger$$
$$= U \cdot \mathrm{diag}(f(\lambda_i)) \cdot U^{-1} = U \begin{pmatrix} f(\lambda_1) & & \\ & f(\lambda_2) & \\ & & \ddots \end{pmatrix} U^{-1}$$

$$e^{\mathrm{Tr}\, M} = \exp(\mathrm{Tr}(U^{-1}DU)) = U \exp\left(\mathrm{Tr}\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}\right) U^{-1}$$

$= U e^{\lambda_1 + \lambda_2 + \dots} U^{-1} = U \,\mathrm{Det}\, e^{D} U^{-1}$

$= \det e^{M}$

* Characterization of eigensystems:

- note (SVD): $M = UWV^T$ but can we say $M = UDU^{-1}$?

a) End(n): any $n \times n$ matrix has n complex eigenvalues
however degenerate λ_i may not have have same # of eigenvectors
defective eigenvalues $\approx$ dilation + nilpotent parts of matrix

Example: Shear: $M = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$Z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad Z\vec{v}_0 = 0\vec{v}_0$ $\quad = 1\, I + a\, Z$

$Z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad Z\vec{v}_1 = \vec{v}_0$ $\leftarrow$ Jordan chain of generalized eigenvectors.

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Thus defective matrices still admit Jordan decompositions
$M = UJU^{-1}$

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 & & & & \\ 0 & \lambda_1 & 1 & & & & \\ 0 & 0 & \lambda_1 & & & & \\ & & & \lambda_1 & 1 & & \\ & & & 0 & \lambda_1 & & \\ & & & & & \lambda_1 & \\ & & & & & & \lambda_2 \end{pmatrix}$$

block diag w/ superdiag. 1's.

b) Normal matrices $N^{\dagger}N = NN^{\dagger}$

equivalently $N = H + iK$ $\quad H^{\dagger} = H$ $\quad K^{\dagger} = K$ $\quad [H,K] = 0$

H and K each have real eigenvalues

since $[H,K] = 0$, there exist common eigenvectors

thus N has a set of "independent" complex eigenvalues $\quad N\vec{u}_i = \lambda_i \vec{u}_i \quad \lambda_i = x_i + i y_i$

$N^{\dagger}$ has eigenvalues $\quad N^{\dagger}\vec{u}_i = \lambda_i^* \vec{u}_i \quad \lambda_i^* = x_i - i y_i$

The eigenvectors are orthogonal $\vec{u}_i \cdot \vec{u}_j = 0$ if $\lambda_i \neq \lambda_j$

so N has a unitary diagonalization

$N = UDU^{\dagger}$ where $U^{\dagger}U = I$

Normal matrix analogy lists the special cases:

Hermitian: real eigs; antiHermitian: imaginary
Positive def: positive eigs Unitary: units $|\lambda_i|=1$

- Proof of simultaneous diagonalization:

if $A\vec{v}=\lambda\vec{v}$ $AB=BA$

then $A(B\vec{v}) = B\lambda\vec{v} = \lambda(B\vec{v})$

thus $B\vec{v} \in \lambda$ eigenspace Λ of A, λ

diagonalize B on Λ $B\vec{u}=\mu\vec{u}$

then $A\vec{u}=\lambda\vec{u}$ also.

- Proof of conjugate eigenvalues & unitary eigenvectors:

if $N\vec{u}_i=\lambda_i\vec{u}_i$ and $N^\dagger\vec{u}_i=\mu\vec{u}_i$ then

$u_i^\dagger N^\dagger N u_j = (\lambda_i^* \lambda_j = \mu_i \lambda_j)\, u_i^\dagger u_j$

if $i=j$ then $\lambda_i^*=\mu_j$

if $\lambda_i \neq \lambda_j$ then $u_i^\dagger u_j = 0$