L13''-Commutator and Simultaneous Diagonalization

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* The importance of being... Earnest?

a) Unitary (orthogonal): simple inversion!

- if $U^\dagger U = I$ then $U^{-1} = U^\dagger$
  eg. component transformations $\mathbb{v}' = U\mathbb{v}$ $\mathbb{v} = U^{-1}\mathbb{v}'$
  similarity (matrix element) transformations $A = UDU^{-1}$
- especially important for infinite-dimensional operators!

b) Hermitian (symmetric): observables!

- if $H^\dagger = H$ then $H = UDU^\dagger$ $D^* = D$ $U^\dagger U = 1$
  real eigenvalues (measurements) & orthogonal eigenstates

c) Diagonal: practical consideration: matrix multiplication

- $A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} = B + A$ elementwise addition

BUT:

- $AB \neq BA$ in general (non commutative)
- $f(A) \neq \begin{pmatrix} f(a_{11}) & f(a_{21}) \\ f(a_{12}) & f(a_{22}) \end{pmatrix}$ ie. $A^2 \neq \begin{pmatrix} a_{11}^2 & a_{12}^2 \\ a_{21}^2 & a_{22}^2 \end{pmatrix}$ elements swish around!

HOWEVER:

- $A'B' = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \begin{pmatrix} b_1 & & \\ & b_2 & \\ & & b_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 & & \\ & a_2b_2 & \\ & & a_3b_3 \end{pmatrix} = B'A'$ for diagonal matrices!
- if $f(x) = \sum_i f_i x^i$ then $f(A) \equiv \sum_i f_i A^i$ (Taylor Series)
  so $f(A') = \begin{pmatrix} f(a_1) & & \\ & f(a_2) & \\ & & \ddots \end{pmatrix}$
- if $A = UDU^{-1}$ then $A^2 = UD\underbrace{U^{-1}U}_{=I}DU^{-1} = UD^2U^{-1}$

$$f(A) = \sum_{i=0}^{\infty} f_i (UDU^{-1})^i = U^{-1}\left(\sum_{i=0}^{\infty} f_i D^i\right) U^{-1} = U f(D) U^{-1}$$

d) Commutative: $AB = BA$ Simultaneous measurements!

- physical measurements are represented by the eigenvalues.
- if A and B are diagonal, then $AB = BA$ or the commutator $[A,B] \equiv AB - BA$ equals 0. and both A, B have definite measurements for the basis states $\hat{e}_1, \hat{e}_2, \ldots$ (canonical basis)
- is the converse true: if $[A,B]=0$ then they can be simultaneously diagonalized? YES!
  $A = UD_AU^{-1}$ and $B = UD_BU^{-1}$ for some U
- thus the commutator is strongly connected to the Heisenberg Uncertainty Principle
  $$[\hat{x},\hat{p}]\,\psi(x) = (x(-i\hbar\partial_x) - (-i\hbar\partial_x)x)\,\psi(x) = 1\cdot\psi(x)$$
  $$\underbrace{x\psi + \tfrac{\partial x}{\partial x}\psi}_{\text{product rule:}}$$
  $\Rightarrow p, x$ complementary!

e) Normal: $[N, N^\dagger] = 0$ complex-matrix analogy.

let $H = \frac{1}{2}(N + N^\dagger)$ so that $H^\dagger = H$ and $N = H + iK$

$K = \frac{1}{2i}(N - N^\dagger)$ $K^\dagger = K$ $N^\dagger = H - iK$

these are the "real" and "imaginary" parts of N

note they both have complex matrix elements!

$$[N, N^\dagger] = [H + iK, H - iK] = \cancel{[H,H]}^{\,0} - i[H,K] + i[K,H] + \cancel{[K,K]}^{\,0}$$
$$= -2i[H,K] \quad \text{so } N, N^\dagger \text{ commute iff } H, K \text{ do}$$

then $H_D = U^\dagger H U = \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & \ddots \end{pmatrix}$ $K_D = U^\dagger K U = \begin{pmatrix} k_1 & & \\ & k_2 & \\ & & \ddots \end{pmatrix}$

thus $D = U^\dagger(H + iK)U = \begin{pmatrix} h_1 + ik_1 & & \\ & h_2 + ik_2 & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} n_1 & & \\ & n_2 & \\ & & n_3 \ddots \end{pmatrix}$ or $N = UDU^\dagger$

N behaves like n independent complex numbers!

* classification of normal matrices:

rect.				polar			
$H^\dagger = H$	$n_i \in \mathbb{R}$	Hermitian		$P^\dagger = P$	and $n_i > 0$	Positive definite	
$T^T = T \in \mathbb{R}^{n\times n}$	"	Symmetric		$S^T = S$	"	"	
sym / antisym							
$K^\dagger + K = 0$	$i n_i \in \mathbb{R}$	anti Hermitian	$\mathrm{Tr} = e^{i\phi}$	$U^\dagger U = 1$	$n_i = e^{i\phi_i}$	Unitary	$\lvert\mathrm{Det}\rvert = 1$
$A^T + A = 0$	$\pm i n_i$	antisymmetric	$\mathrm{Tr} = 0$	$V^T V = 1$	$n_i = e^{\pm i\phi_i}$	Orthogonal	$\mathrm{Det} = \pm 1$

* Simultaneous diagonalization theorem:

If $U^{-1}AU = D$ is diagonal and $[A, B] = 0$,
then $U^{-1}BU$ is block diagonal over the direct sum of eigenspaces of A (with λ_i)

proof: let $A\vec{u}_i = \lambda \vec{u}_i$ then $A(B\vec{u}_i) = BA\vec{u}_i = \lambda(B\vec{u}_i)$
thus $B\vec{u}_i$ is also an eigenvector of A w/ λ.
Therefore, B maps the eigenspace of A, λ into itself.

$$A\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix} = \begin{pmatrix}\lambda_1 I & 0 & 0\\ 0 & \lambda_2 I & 0\\ 0 & 0 & \lambda_3 I\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix} \qquad B\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix} = \begin{pmatrix}B_1 & 0 & 0\\ 0 & B_2 & 0\\ 0 & 0 & B_3\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix}$$

Note: you can further diagonalize each block B_i without destroying the diagonalization of A, since A_i is just a multiple of I and $U^{-1}IU = I$ still.