L13-Hilbert spaces

Monday, September 28, 2015 9:20 AM

* Hilbert space – Cauchy sequences converge.
  countable not dense
  $\sum_{i=1}^{\infty} |n\rangle a_i$ instead of $\int dx\, |x\rangle f(x)$

  need well-defined inner product
  $\delta(x-x')$ not good enough

3) orthonormality & closure, matrix elements.

NOTATION vs. PRINCIPLES.

Hilbert Space

exp. INDEP./COMPL. $\underbrace{\vec{v}}_{\text{vector}} = \underbrace{\sum_i}_{\text{sum dummy index}} \underbrace{\hat{e}_i}_{\text{basis}} \underbrace{v_i}_{\text{component}}$ $\quad v = \hat{e}\, v \quad |f\rangle = \int dx\, |x\rangle f(x) \quad = \sum_{n=1}^{\infty} |n\rangle f_n$

natural basis $\hat{e}_i = \sum_j \hat{e}_i \delta_{ij}$ def'n. $\quad \hat{e} = \hat{e}\,\mathbb{I} \quad |x\rangle = \int dx'\, |x'\rangle \delta(x-x')$ defn of δ_{ij} $\quad = \sum_{n=1}^{\infty} |n\rangle \delta_{mn}$

ORTHO. $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ → indep. $\quad \hat{e}^T \cdot \hat{e} = \mathbb{I} \quad \langle x | x' \rangle = \delta(x-x') \quad \langle n | m \rangle = \delta_{nm}$

(matrix)

inner product $\vec{v} \cdot \vec{w} = \sum_i v_i^* w_i \quad = v^\dagger w \quad \langle f | g \rangle = \int dx\, f^*(x)\, g(x)$ Follows orthonormality OR closure! $\quad \langle f | g \rangle = \sum_n f_n^* g_n$

extract comp. $\hat{e}_i \cdot \vec{v} = v_i \quad \hat{e}^T \cdot \vec{v} = v \quad \langle x | f \rangle = f(x) \quad \langle n | f \rangle = f_n$

adjoint $\vec{v}\cdot = \sum_i v_i^* \hat{e}_i \cdot \quad \langle f | = \int dx\, f^*(x) \langle x |$

CLOSURE projectors $\sum_i \hat{e}_i \hat{e}_i \equiv \sum_i P_i = \mathbb{I} \quad \hat{e}\hat{e}^T \cdot = 1 \quad \int dx\, |x\rangle\langle x| = 1 \quad \sum_n |n\rangle\langle n| = 1$

operator $\vec{w} = M(\vec{v}) \quad w = \underset{\sim}{M}\, v \quad |g\rangle = M|f\rangle \quad g(x) = \underbrace{\int dx\, M(x,x') f(x)} \quad g_m = \sum_n M_{mn} f_n$

components matrix elements $\hat{e}_i \cdot \vec{w} = \underbrace{\hat{e}_i \cdot M(\sum_j \hat{e}_j)}\, \hat{e}_j \cdot \vec{v} \quad \langle x | g \rangle = \underbrace{\langle x | M | \int dx' |x'\rangle}\, \langle x' | f \rangle \quad \langle m | g \rangle = \underbrace{\langle m | M \sum_n |n\rangle}\langle n | f \rangle$

change of basis $\hat{e}'_i = \sum_j \hat{e}_j\, \underbrace{\hat{e}_j \cdot \hat{e}'_i} = \sum_j \hat{e}_j\, \underbrace{R_{ji}}$ $\quad \hat{e}' = \hat{e} R \quad |k\rangle = \int dx\, |x\rangle\underbrace{\langle x | k\rangle} = \int dx\, |x\rangle \underbrace{\frac{1}{\sqrt{2\pi}} e^{ikx}} \quad \langle k | s \rangle = \phi(k) \quad \langle x | s \rangle = \psi(x)$

ORTHO NEW BASIS orthogonality unitarity. $\sum_k R^\dagger_{ik} R_{kj} = \delta_{ij} \quad R^\dagger R = \mathbb{I} \quad \langle k' | k \rangle = \delta(k-k') \quad \int dx \frac{1}{2\pi} e^{i(k-k')x} = \delta(k-k')$

CLOSURE NEW BASIS $\sum_i R_{ki} R^\dagger_{ij} = \delta_{ij} \quad R R^\dagger = \mathbb{I} \quad \int dk\, |k\rangle\langle k| = 1 \quad \int dk \frac{1}{2\pi} e^{ik(x-x')} = \delta(x-x')$

IDENTITY TRANSFORM change of co-ordinates. $\hat{e}_i \cdot \vec{v} = \sum_j \hat{e}_i \cdot \hat{e}_j\, \hat{e}_j \cdot \vec{v} = \sum_j R_{ij} v_j \quad R v' = v$
$\langle x | s \rangle = \int dk\, \langle x | 1 | k \rangle \langle k | s \rangle \quad \psi(x) = \int dk \frac{1}{\sqrt{2\pi}} e^{ikx} \phi(x)$
$\langle k | s \rangle = \int dx\, \langle k | 1 | x \rangle \langle x | s \rangle \quad \phi(k) = \int dx \frac{1}{\sqrt{2\pi}} e^{-ikx} \psi(x)$

* proofs

ortho → indep. (not nec. completeness)
if $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ and $\alpha_i e_i = 0$
then $e_j \cdot \alpha_i e_i = \alpha_i \delta_{ij} = \alpha_j = 0 \rightarrow \alpha_j = 0$

ortho to component dot
$\vec{v} \cdot \vec{w} = \sum v_i^* \hat{e}_i \cdot \sum_j \hat{e}_j w_j = \sum_{ij} v_i^* \delta_{ij} w_j = \sum_i v_i^* w_i$

components of ortho
$\delta_{ij} \delta_{jk} = \delta_{ik} \quad \int dx'\, \delta(x-x') \delta(x''-x') = \delta(x-x'')$

$$v \cdot w = \sum v_i^* e_i \cdot \sum_j e_j w_j = \sum_{ij} v_i^* \delta_{ij} w_j = \sum_i v_i^* w_i$$

components of ortho

$$\delta_{ij}\delta_{kj} = \delta_{ik} \qquad \int dx'\, \delta(x-x')\,\delta(x''-x') = \delta(x-x'')$$

$$\mathbb{e}^T \cdot \mathbb{e} = \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \end{pmatrix} \cdot (\hat{e}_1\ \hat{e}_2 \cdots) = I$$

extract components:

$$\hat{e}_i \cdot \vec{v} = e_i \cdot \sum e_j v_j = \delta_{ij} v_j = v_i \qquad \langle x|f\rangle = \langle x| \int dx'\, |x'\rangle f(x') = \int dx'\, \delta(x-x') f(x')$$

completeness & orthonormality $\Rightarrow$ closure.

$$\vec{v} = \sum_i \hat{e}_i v_i = \overbrace{\sum_i \underbrace{\hat{e}_i \hat{e}_i}_{P_i}}^{I} \cdot \vec{v} \qquad \text{inv. of } \hat{\mathbb{e}} \cdot \hat{\mathbb{e}} = \mathbb{I}$$

closure $\Rightarrow$ completeness & ortho?

$$\vec{v} = \sum_i \hat{e}_i \hat{e}_i \cdot \vec{v} \quad \text{thus } v_i \equiv \hat{e}_i \cdot \vec{v} \qquad \hat{e}_i = \sum_j \hat{e}_j\, \hat{e}_j \cdot \hat{e}_i \Rightarrow \text{thus } \hat{e}_j \cdot \hat{e}_i = \delta_{ij}\ ?$$

closure $\Rightarrow$ components of $\cdot$

$$\vec{v} \cdot \vec{w} = \vec{v} \cdot \sum_i \hat{e}_i \hat{e}_i \cdot \vec{w} = \sum_i v_i^* w_i$$