

* Dirac notation for operators on Hilbert spaces

- important "tools" for going back and forth:
 - adjoint: $\langle f |^\dagger \equiv | f \rangle^\dagger \quad z^\dagger \equiv z^* \quad \langle f | H^\dagger | g \rangle = \langle g | H | f \rangle^*$
 - orthonormality: $\langle n | m \rangle = \delta_{nm} \quad \langle x | x' \rangle = \delta(x-x')$
 - closure: $\sum_{n=0}^{\infty} | n \rangle \langle n | = 1 \quad \int dx | x \rangle \langle x | = 1$
 - components: $\langle n | f \rangle = f_n \quad \langle x | f \rangle = f(x) \quad \langle x | n \rangle = \phi_n(x)$
 - matrix elements: $\langle m | H | n \rangle = H_{mn} \quad \langle x | H | x' \rangle = H(x-x')$

- what is the "matrix" of an ω -d operator?

$$\begin{aligned}
 | g \rangle &= H | f \rangle \\
 \langle m | g \rangle &= \langle m | H \sum_n | n \rangle \langle n | | f \rangle \\
 g_m &= \sum_n H_{mn} f_n
 \end{aligned}
 \quad
 \begin{pmatrix} g_1 \\ g_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & \dots \\ H_{21} & H_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}$$

$$\begin{aligned}
 \langle x | g \rangle &= \langle x | H \int dx' | x' \rangle \langle x' | f \rangle \\
 g(x) &= \int dx' \underbrace{H(x, x')}_{\text{kernel of integral transform}} f(x')
 \end{aligned}$$

- example: identity transform $1 | f \rangle = | f \rangle$

$$\begin{aligned}
 \langle x | f \rangle &= \int dx' \langle x | 1 | x' \rangle \langle x' | f \rangle \\
 f(x) &= \int \delta(x-x') f(x')
 \end{aligned}
 \quad
 \begin{aligned}
 & * \text{ what does matrix} \\
 & \text{look like?}
 \end{aligned}$$

$$\begin{aligned}
 \langle x | f \rangle &= \int dk \langle x | 1 | k \rangle \langle k | f \rangle \\
 \tilde{f}(k) &= \int dx e^{ikx} f(x)
 \end{aligned}
 \quad
 \begin{aligned}
 & * \text{ matrix?}
 \end{aligned}$$

- example: derivative operator * matrix?

$$f'(x) = \int \delta(x-x') \frac{d}{dx'} f(x') dx' = - \int f(x') \frac{d}{dx'} \delta(x-x') dx'$$

$$\text{so } \langle x | \frac{d}{dx} | x' \rangle = -\delta'(x-x')$$

* Hermitian (symmetric) operators: spectrum

- any matrix has n complex eigenvalues (FTA)
- some "defective" matrices have fewer eigenvectors

$$D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \rightarrow J = \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_2 & 1 \\ & & & \lambda_2 \end{pmatrix} \quad \text{Jordan blocks}$$

- Hermitian matrices $H^\dagger = H$ are Hermitian in any orthogonal basis: if $M = U^\dagger H U$ then $M^\dagger = (U^\dagger H U)^\dagger = U^\dagger H^\dagger U = U^\dagger H U = M$

① Thus Hermitian matrices are NOT defective

$$\text{let } H \vec{u}_i = \lambda_i \vec{u}_i \quad \text{then } \vec{u}_i^\dagger H^\dagger = \lambda_i^* \vec{u}_i^\dagger$$

$$\lambda_j^* \vec{u}_j^\dagger \vec{u}_i = \vec{u}_j^\dagger H \vec{u}_i = \vec{u}_j^\dagger \vec{u}_i \lambda_i$$

② If $i = j$, $\lambda_i^* = \lambda$ H has real eigenvalues!

If $\lambda_i = \lambda_j$ then any $\alpha \vec{u}_i + \beta \vec{u}_j$ is an eigenvector

③ If $\lambda_i \neq \lambda_j$, $\vec{u}_j^\dagger \vec{u}_i = 0$ H has orthogonal eigenspaces!

* Example: the operator $D \sim \frac{d}{dx}$ is antiHermitian

$$\langle g | D | f \rangle \equiv \int dx \, g(x) \frac{d}{dx} f(x) = \int g \, df$$

$$= \underbrace{fg}_{\text{somehow!}} - \int f dg = - \langle f | D | g \rangle$$

Thus the operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$ is Hermitian.

* Application: Sturm Liouville theory of ODE's

Given a 2nd order differential operator L

$$L[y] = p_2(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_0(x)$$

there exists an integrating function

$$w(x) = \frac{1}{p_2(x)} \exp\left(\int_0^x \frac{p_1(x')}{p_2(x')} dx'\right) \text{ such that}$$

$$L = \frac{1}{w(x)} \left(\frac{d}{dx} p(x) \frac{d}{dx} - q(x) \right) \text{ is self-adjoint } L^\dagger = L$$

with respect to the inner product

$$\langle y_2 | y_1 \rangle = \int_a^b w(x) dx y_2^*(x) y_1(x)$$

$$\text{if } w(x) y_2^*(x) y_1(x) \Big|_a^b = 0 \text{ (boundary conditions)}$$

Thus $L = \sum_{n=1}^{\infty} \lambda_n |n\rangle \langle n|$ has a complete

set of orthogonal eigenfunctions, since

$$\lambda_2^* \langle y_2 | y_1 \rangle = \langle y_2 | L | y_1 \rangle = \langle y_2 | y_1 \rangle \lambda_1$$

which is the same as above for matrix H .