L16-SHO Operator Algebra

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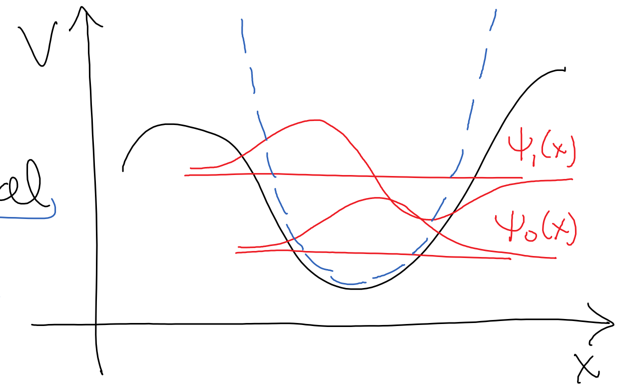
* one of the most important potentials: periodic motion is usually $\underbrace{\text{vibrational}}_{\text{approximated by harmonic potential}}$, or $\underbrace{\text{rotational}}_{\text{rigid body rotor - ch4}}$



$$V(x) = \underbrace{\cancel{V(x_0)}^{\,0}}_{\text{irrelevant}} + V'(x_0)\underbrace{(x-x_0)}_{\text{see HO6}} + \tfrac{1}{2}V''(x_0)\underbrace{(x-x_0)^2}_{\text{SHO}} \rightarrow \tfrac{1}{2}m\omega^2 x^2$$

* classical solution: $F_{ext} = m\ddot{x} + b\dot{x} + kx$ let $x = x_0 e^{i\omega t}$

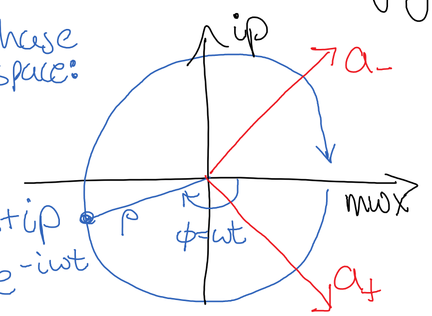
if $F_{ext} = 0$ then $-m\omega^2 + ib\omega + k = 0$ $\quad \omega = \frac{ib}{2m} \pm \sqrt{\left(\frac{ib}{2m}\right)^2 + \frac{k}{m}}$

if $b=0$ (undamped) then $k = m\omega^2$, where ω = classical frequency

* algebraic method: (also ang. momentum & SUSY potentials!) factorize the potential into a product of canonical conjugates

$$H = \frac{1}{2m}\left(p^2 + (m\omega x)^2\right) = (\overbrace{iu+v}^{\sim a_-})(\overbrace{-iu+v}^{\sim a_+}) = |\rho|^2$$

Phase space:



classical: $T = u^2 = \frac{p^2}{2m}$ $\quad V = v^2 = \frac{1}{2}m\omega^2 x^2$

$m\omega x + ip = \rho e^{-i\omega t}$

let $a_\pm = \frac{1}{\sqrt{\hbar\omega}}(\mp iu + v) = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$

* note: Dirac did similar to $E^2 + p^2 = m^2$ to get the Dirac equation.

but $H \neq a_- a_+$ quantum mechanically because x, p don't commute

* cannonical commutation relationship: $\boxed{[x,p] = i\hbar}$

$$\underline{[x,p]\psi(x)} = (xp - px)\psi(x) = [x(-i\hbar\partial_x) - (-i\hbar\partial_x)x]\psi(x)$$
$$= i\hbar[-x\partial_x\psi + \partial_x(x\psi)] = i\hbar(-x\psi' + \psi + x\psi') = \underline{i\hbar}\,\psi(x)$$

$$[a_-, a_+] = \frac{1}{2\hbar m\omega}[ip + m\omega x, -ip + m\omega x] \qquad [p,p] = [x,x] = 0$$
$$= \frac{1}{i\hbar}[x,p] = 1 = -[a_+, a_-] \qquad [p,x] = -[x,p] = i\hbar$$

thus $a_- a_+ = \frac{1}{2\hbar m\omega}(ip + m\omega x)(-ip + m\omega x)$
$$= \frac{1}{2\hbar m\omega}\left(p^2 + (m\omega x)^2 - im\omega[p,x]\right) = \mathcal{H}/\hbar\omega + 1/2$$

$$\boxed{\mathcal{H} = \hbar\omega\left(a_- a_+ - \tfrac{1}{2}\right) = \hbar\omega\left(a_+ a_- + \tfrac{1}{2}\right)} \qquad [a_-, a_+] = 1$$

* product rule – acts similar to a derivative !

$$[a, bc] = (abc - bca) = (abc \underbrace{- bac + bac}_{\text{insert these for symmetry.}} - bca)$$

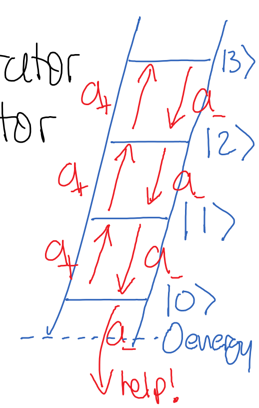
$$\boxed{[a, bc] = [a,b]c + b[a,c]} \quad \text{compare: } \partial_a(bc) = (\partial_a b)c + b(\partial_a c)$$

thus $[\mathcal{H}, a_\pm] = \hbar\omega[a_+ a_-, a_\pm] = \hbar\omega\left(\underbrace{[a_+, a_\pm]}_{0 \text{ or } -1} a_- + a_+ \underbrace{[a_-, a_\pm]}_{1 \text{ or } 0}\right) = \pm\hbar\omega a_\pm$

* eigenvalues: let $\mathcal{H}|n\rangle = E_n|n\rangle$, use $[\mathcal{H}, a_\pm] = \pm\hbar\omega a_\pm$

then $\mathcal{H} a_\pm|n\rangle = (a_\pm \mathcal{H} \pm \hbar\omega a_\pm)|n\rangle = (E_n \pm \hbar\omega) a_\pm|n\rangle$
and $a_\pm|n\rangle$ is a different eigenstate with higher/lower energy

thus $a_\pm$ are called ladder operators
$a = a_-$ is called lowering or annihilation operator
$a^\dagger = a_+$ is called raising or creation operator



let $|0\rangle$ be the ground state (lowest energy)
then $|n\rangle = A_n a_+^n|0\rangle \qquad a_-\underbrace{|0\rangle}_{\text{unit vector}} = 0$ ← origin: 0 vector

$$\mathcal{H}|0\rangle = \hbar\omega(\underbrace{a_+ a_-}_{n} + \tfrac{1}{2})|0\rangle = \tfrac{1}{2}|0\rangle \quad \text{thus } E_n = \hbar\omega(n + \tfrac{1}{2})$$

* normalization: let $a_+|n\rangle = c_n|n+1\rangle \quad a_-|n\rangle = d_n|n-1\rangle$

note: $a_+a_-|n\rangle = \mathcal{H}/\hbar\omega - \frac{1}{2}|n\rangle = n|n\rangle \qquad a_-a_+|n\rangle = n+1|n\rangle$

$$n = \langle n|n|n\rangle = \langle n|\overleftarrow{a_+}\,\overrightarrow{a_-}|n\rangle = \langle n+1|c_n^* c_n|n+1\rangle = |c_n|^2$$

$$n = \langle n-1|n+1|n-1\rangle = \langle n-1|a_-\,a_+|n-1\rangle = \langle n-1|d_n^* d_n|n-1\rangle = |d_n|^2$$

thus $\boxed{\begin{aligned} a_-|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a_+|n-1\rangle &= \sqrt{n}\,|n\rangle \\ |n\rangle &= \tfrac{1}{\sqrt{n!}}\,a_+^n|0\rangle \end{aligned}}$ $\quad a_+ \sim \begin{pmatrix} 0 & 0 & & & \\ \sqrt{1} & 0 & 0 & & \\ & \sqrt{2} & 0 & 0 & \\ & & \sqrt{3} & 0 & \\ & & & & \ddots \end{pmatrix} \quad a_- \sim \begin{pmatrix} 0 & \sqrt{1} & & & \\ 0 & 0 & \sqrt{2} & & \\ & 0 & 0 & \sqrt{3} & \\ & & 0 & 0 & \ddots \end{pmatrix}$

* in-class: calculate matrices of x, p, $[x,p]$, p^2, x^2, $\mathcal{H}$

* wave functions: $a_-|0\rangle = (i\hat{p} + m\omega x)\,\psi_0(x) = 0$

$$\left(-i^2\hbar\frac{d}{dx} + m\omega x\right)\psi_0(x) = 0 \qquad d\ln\psi_0 = \frac{d\psi_0}{\psi_0} = -\frac{m\omega x}{\hbar}dx = d\,\frac{-m\omega}{2\hbar}x^2$$

$$\psi_0(x) = A_0 e^{-\frac{m\omega}{2\hbar}x^2} \qquad 1 = \int_{-\infty}^{\infty} dx\,|A_0|^2 e^{-\frac{m\omega}{\hbar}x^2} = \sqrt{\frac{\hbar\pi}{m\omega}}$$

$$= \left(\frac{m\omega}{\hbar\pi}\right)e^{-\frac{m\omega}{2\hbar}x^2} \qquad \text{recall L07: } \int_{-\infty}^{\infty} e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

* Summary: key relations:

$$a_\pm = \frac{1}{\sqrt{2\hbar\omega m}}(\mp i\hat{p} + m\omega\hat{x}) = \frac{1}{\sqrt{2}}(\mp\partial_\xi + \xi) \qquad a_-^\dagger = a_+ \qquad [a_-, a_+] = [\partial_\xi, \xi] = 1$$

$$\mathcal{H} = \hbar\omega(\underbrace{a_+a_-}_{n} + \tfrac{1}{2}) \qquad \mathcal{H}|n\rangle = E_n|n\rangle \qquad [a_+a_-, a_\pm] = \pm a_\pm \qquad a_+a_- = n$$

$$n = \langle n|a_+a_-|n\rangle = \langle a_-n|a_-n\rangle \qquad \boxed{\begin{aligned} a_-|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a_+|n-1\rangle &= \sqrt{n}\,|n\rangle \end{aligned}} \qquad a_-|0\rangle = 0$$

$$n+1 = \langle n|a_-a_+|n\rangle = \langle a_+n|a_+n\rangle \qquad \langle\xi|0\rangle = e^{-\frac{1}{2}\xi^2}$$