L17-SHO Frobenius Method

Monday, October 19, 2015 07:05

* TISE for SHO : $\hat{H}\Psi = E\Psi$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right)\Psi(x) = E\,\Psi(x)$$

a) solve ODE
b) apply B.C.'s.

let $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x \quad K = \frac{2E}{\hbar\omega} \quad \Rightarrow \quad \frac{d^2\Psi}{d\xi^2} + (\xi^2 - K)\Psi = 0$

- asymptotic limit: if $\xi^2 \gg K$ then $\frac{d^2\Psi}{d\xi^2} = -\xi^2\Psi$, $\Psi \approx e^{-\xi^2/2}$

smaller   dominant as $\xi \to \infty$

$$\frac{d}{d\xi}\left(\frac{d}{d\xi}e^{-\xi^2/2}\right) = \frac{d}{d\xi}\left(-\xi e^{-\xi^2/2}\right) = \cancel{-1}\cdot e^{-\xi^2/2} + (-\xi)(-\xi)e^{-\xi^2/2} \approx \xi^2 e^{-\xi^2/2}$$

let $\Psi(\xi) = h(\xi)\,e^{-\xi^2/2}$ to factor out this dependence

$$\Psi' = h' e^{-\xi^2/2} + h(-\xi)e^{-\xi^2/2}$$
$$\Psi'' = h'' e^{-\xi^2/2} + 2h'(-\xi)e^{-\xi^2/2} - h e^{-\xi^2/2} + h\,\xi^2 e^{-\xi^2/2}$$
$$\Psi'' - (\xi^2 - K)\Psi = \left(h'' - 2h'\xi + h\xi^2 + (K-1)h\right)e^{-\xi^2/2} = 0$$

thus $h'' - 2\xi h' + (K-1)h = 0$ Hermite ODE with $K+1 \to 2n$

- Power series solution: let $h = \sum_{j=0}^{\infty} a_j \xi^j \quad h' = \sum_{j=0}^{\infty} a_j\, j\, \xi^{j-1}$

plug these into ODE: $\quad h'' = \sum_{j=0}^{\infty} a_j\, j(j-1)\,\xi^{j-2} = \sum_{j=0}^{\infty} (j+1)(j+2)\,a_{j+2}\,\xi^j$

$$\sum_{j=0}^{\infty}\left[(j+1)(j+2)\,a_{j+2} - 2j\,a_j + (K-1)\,a_j\right]\xi^j = 0 \qquad a_{j+2} = \frac{2j+1-K}{(j+1)(j+2)}\,a_j$$

solution: $h(\xi) = \left[h_{even} = \underbrace{a_0}_{\text{normalization}} + a_2\xi^2 + a_4\xi^4 + \dots\right] + \left[h_{odd} = \underbrace{a_1}_{\text{norm}}\xi + a_3\xi^3 + \dots\right]$

by recursion   recursion

- quantization: $a_{j+2} \approx \frac{1}{j/2}\,a_j \approx \frac{a_0}{(j/2)!}$ exponential growth!

$h(\xi) \approx a_0 \sum^{\infty} \frac{1}{(j/2)!}\xi^j \approx a_0 \sum \frac{1}{j!}\xi^{2j} \approx a_0\, e^{+\xi^2}$ blows up.

only get normalized solution if series terminates

if $K_n = 2n+1$ then $a_{n+2} = a_{n+4} = \dots = 0$. K_n satisfies B.C.

thus $E_n = \frac{\hbar\omega}{2}(2n+1) = \hbar\omega(n+\frac{1}{2})$ $\quad a_{j+2}^{(n)} = \frac{-2(n-j)}{(j+1)(j+2)}$

- Hermite polynomials: "normalized" so $a_n^{(n)} = 2^n$

$h_0(\xi) = a_0^{(0)} = 1$ $\quad \boxed{H_1(\xi) = 1}$ $\quad h_1(\xi) = a_1^{(1)} = \xi$ $\quad \boxed{H_2(\xi) = 2\xi}$

$h_2(\xi) = a_0^{(2)} + a_2^{(2)}\xi^2 = 1-2$ $\quad h_3(\xi) = a_1^{(3)}\xi + a_3^{(3)}\xi^3 = \xi - \frac{2}{3}\xi^3$

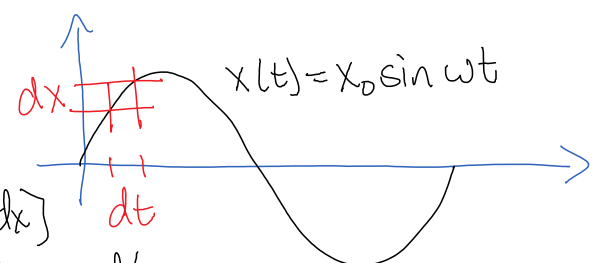
$\boxed{H_2(\xi) = 4\xi^2 - 2}$ $\quad \boxed{H_3(\xi) = 8\xi^3 - 12\xi}$

- Normalized wave functions:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

$$\int_{-\infty}^{\infty} \psi_n(x)\psi_m(x)\,dx = \frac{1}{\sqrt{\pi\, 2^n n!\, 2^m m!}} \int_{-\infty}^{\infty} \underbrace{e^{-\xi^2}}_{\text{weight } w(\xi)} d\xi\, H_n(\xi) H_m(\xi) = \delta_{nm}$$

- Classical probability density:



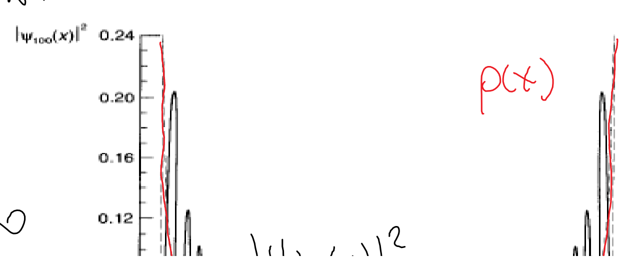
$\rho\,dx$ = probability that $x \in [x, x+dx]$
$\quad \propto$ time spent in this region dt.

thus $\rho(x) \propto \frac{1}{dx/dt} = \frac{1}{\cos(\omega t)} = \frac{1}{\sqrt{1-\sin^2\omega t}} = \frac{1}{\sqrt{1-(x/x_0)^2}}$

$$\int_{-x_0}^{x_0} \rho(x)\,dx = \int_{-x_0}^{x_0} \frac{k\,dx}{\sqrt{1-(x/x_0)^2}} = k x_0 \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = k x_0 \frac{\pi}{2} = 1 \qquad u = x/x_0$$

thus $\rho(x) = \frac{2/\pi}{\sqrt{x_0^2 - x^2}}$

Compare with $|\psi_{100}(x)|^2$ to



Compare with $|\psi_{100}(x)|^2$ to see classical limit.

(Griffiths)

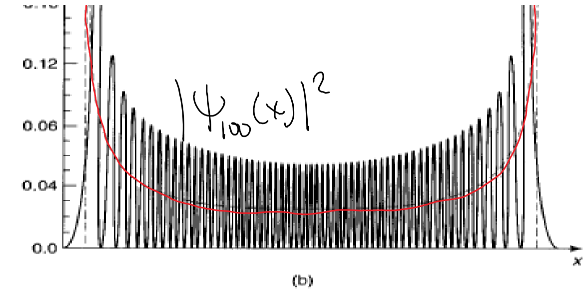


Figure 2.5: (a) The first four stationary states of the harmonic oscillator. (b) Graph of $|\psi_{100}|^2$, with the classical distribution (dashed curve) superimposed.