L18-Delta function potential

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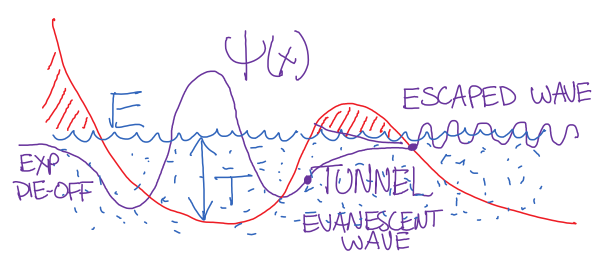
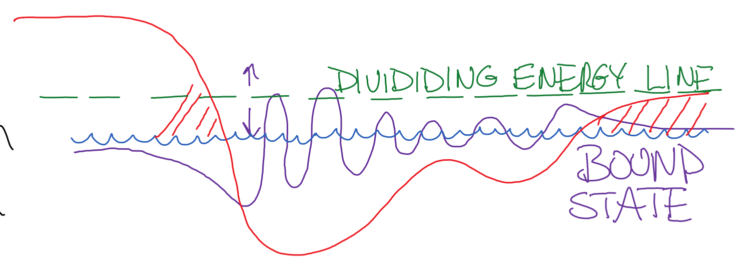
* Concepts

- **turning point**: $V(x_{TP}) = E \Rightarrow T(x_{TP}) = 0$
  - classical particle turns around
  - quantum wave: curvature $= k^2$ switches from oscillatory: $k^2\psi < 0$ to exponential $k^2\psi > 0$

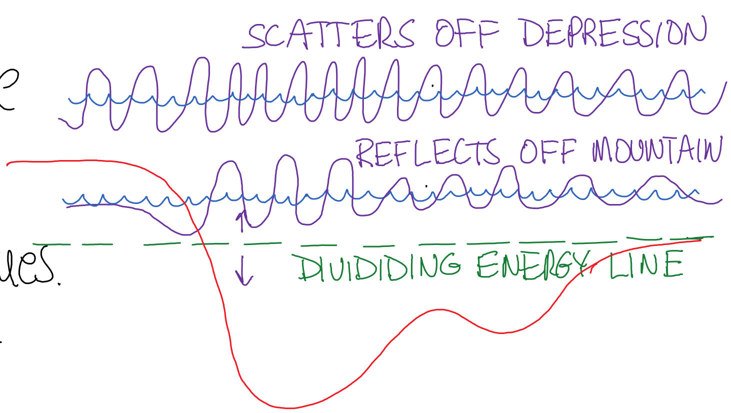
- **tunnelling**: (evanescent wave) exponential dropoff through a mountain. oscillatory with reduced amplitude on other side.



- **bound state**: exponentially decays at both ends.
  - quantized by external B.C.'s
  - Sturm-Liouville system
  - discrete energy spectrum
  - normalizeable



- **scattering state** can escape to one or both ends
  - missing external B.C.'s
  - still has eigenfunctions/values.
  - continuous energy spectrum
  - not normalizeable



- **boundary conditions**

a) external conditions $x \to \pm\infty$ require normalization of bound states $\to$ quantization

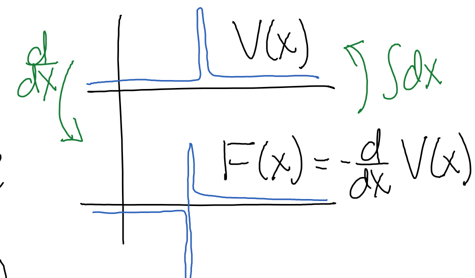
$|\psi(x)| < k x^{-1}$ as $x \to \pm\infty$ so that $\int_{-\infty}^{\infty} dx |\psi(x)|^2 = 1$

- impossible for scattering states; require
  $\langle k | k' \rangle \equiv \int_{-\infty}^{\infty} dx\, \psi_k^*(x)\, \psi_{k'}(x) = \delta(k-k')$ for convenience
  (wavefunctions still normalizable using $\int dk\, A(k)\, \psi_k(x)$)

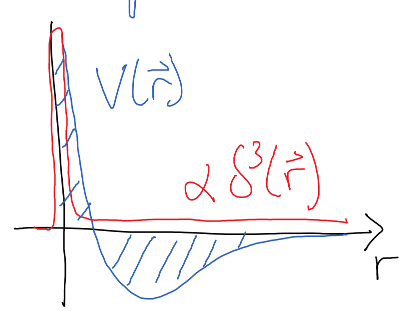
b) internal conditions: continuity of DE accross boundary.
- just like E&M: $\int_{-\varepsilon}^{\varepsilon} \nabla \cdots = \hat{n} \Delta$ $\quad \int_{-\varepsilon}^{\varepsilon} \underbrace{q\, \delta(x)}_{\rho(x)} = q$
  i) $\Delta \psi |_\partial = 0$ $\quad$ ii) $\Delta \psi' |_\partial = \alpha \psi |_\partial$ $\quad \alpha \equiv -\int_{-\varepsilon}^{\varepsilon} dx V(x)$

- Dirac δ-function – the "un" distribution
- not a function because it's properties transcend $f: \mathbb{R} \to \mathbb{R} \cup \{\infty\}$
- it is a distribution (measure, density, functional, differential form) because it "lives inside an integral" $\int dx\, \delta(x-a)\, f(x) \equiv f(a)$
- naturally a ⟨bra| not a |ket⟩ $\quad \langle x| \sim \int dx' \delta(x'-x) \ldots$
- simplest definition: $\delta(x)\, dx \equiv d\,\Theta(x)$ differential of Heaviside step fn

* Dirac δ(x) well $\quad V(x) = -\alpha\, \delta(x)$



- physical significance: localized infinite force
- in three dimensions, Fermi potential captures the volume average of $V(\vec{r})$
  - example: nucleon potential has a "hard core" that deflects others, but also a finite range.
  - $V(r) \sim \alpha\, \delta^3(\vec{r})$ where $\alpha = \int d^3r\, V(\vec{r})$



- discontinuity – must solve ODE in 2 regions
- patch solutions together with continuity equations

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) - \alpha\, \delta(x)\, \psi(x) = E\, \psi(x)$$

– bound state eigenfunctions

if $x \neq 0$: free particle $\frac{d}{dx} e^{\pm \kappa x} = \underbrace{\pm \kappa}_{\text{eigenvalue}} e^{\pm \kappa x}$ $\quad E = -\frac{\hbar^2 \kappa^2}{2m}$

general solution: $\psi_1(x) = \cancel{A}^{\,0} e^{-\kappa x} + B e^{\kappa x}$ $(x<0)$; $\quad \psi_2(x) = F e^{-\kappa x} + \cancel{G}^{\,0} e^{\kappa x}$ $(x>0)$

$\psi_1 \to 0$ as $x \to -\infty$ $\qquad \psi_2 \to 0$ as $x \to +\infty$

external B.C.'s: $\psi_1(x) = B e^{\kappa x}$ $\qquad \psi_2(x) = F e^{-\kappa x}$

internal: $\psi_1(0) = \psi_2(0)$: $B = F$

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} dx \left[\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\,\psi(x) \right]$$

let $\alpha = -\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} dx \underbrace{V(x)}_{-\alpha\,\delta(x)}$

$$\frac{-\hbar^2}{2m} \Delta \psi'(0) - \alpha \psi(0) = 0$$

$$\psi_2'(0) = +\kappa\, \psi_2(0)$$
$$-\quad \psi_1'(0) = -\kappa\, \psi_1(0)$$
$$\Delta \psi'(0) = 2\kappa\ \psi(0)$$

$$\frac{-\hbar^2}{2m} 2\kappa = 2E/\kappa = \alpha$$

one bound state: $\kappa = \frac{-\alpha m}{\hbar^2}$ $\qquad E = \frac{\alpha \kappa}{2} = \frac{-m\alpha^2}{2\hbar^2}$

normalization: $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2\int_0^{\infty} dx\, B^2 e^{-2\kappa x} = \frac{B^2}{\kappa} = 1$

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$$