Scattering and Transfer Matrices

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* δ-function potential (continued)

+ scattering eigenstates

if $x \neq 0$, $\frac{d}{dx} e^{\pm ikx} = \underbrace{\pm ik}_{\text{eigenvalue}} e^{\pm ikx}$ $\quad E = + \frac{\hbar^2 k^2}{2m}$

general solution: $\psi_1(x) = Ae^{ikx} + Be^{-ikx}$ $\quad \psi_2(x) = Fe^{ikx} + Ge^{-ikx}$

this problem is not normalizable – continuous spectrum
external boundary conditions: $A = \text{const.}$ $\quad G = 0$
(amplitudes of incoming waves from left, right)

internal B.C.'s: $\psi_2(0) = \psi_1(0)$: $\quad F + G = A + B$

$$\frac{-\hbar^2}{2m} \Delta \psi'(0) - \alpha \psi(0) = 0 : \quad \frac{-\hbar^2}{2m} ik\left((F-G)-(A-B)\right) = \alpha(A+B)$$

let $\beta = \frac{m\alpha}{\hbar^2 k}$ then $\quad F - G = (1+2i\beta)A - (1-2i\beta)B$

combine these two eq's. into single matrix eq:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1+2i\beta & -1+2i\beta \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

there are 4 unknowns and only 2 equations: we must solve for 2 coefficients in terms of 2 others.

which 2 you solve for depends on the application:

a) transfer matrix: $A,B \to F,G$: useful for "transferring" the amplitudes across multiple boundaries: $A,B \to C,D \to F,G \to \ldots$

b) scattering matrix: $A,G \to B,F$ – physical coefficients for reflection & transmission from the left or right

See Griffiths 1st ed. §2.7 ; 2nd ed problems 2.52, 2.53

* transfer matrix: $\begin{pmatrix} F \\ G \end{pmatrix} = M \begin{pmatrix} A \\ B \end{pmatrix}$ $M = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1+2i\beta & -1+2i\beta \end{pmatrix} = \begin{pmatrix} 1+i\beta & i\beta \\ -i\beta & 1-i\beta \end{pmatrix}$

for multiple scatters, $M_{tot} = M_1 \cdot M_2$ PHY 417, HW5 #2

(recall recursive solution of cylindrical magnetic shields!)

(also recall reflection/transmission coefficients in E&M)

* scattering matrix: $\begin{pmatrix} B \\ F \end{pmatrix} = S \begin{pmatrix} A \\ G \end{pmatrix}$ $S = \frac{1}{M_{22}}\begin{pmatrix} -M_{21} & 1 \\ |M| & M_{12} \end{pmatrix} = \frac{1}{1-i\beta}\begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}$

* Coefficients of: reflection transmission

from: left $(G=0)$ $R_\ell \equiv \left.\frac{|B|^2}{|A|^2}\right|_{G=0} = |S_{11}|^2$ $T_\ell \equiv \left.\frac{|F|^2}{|A|^2}\right|_{G=0} = |S_{21}|^2$

right $(A=0)$ $R_r \equiv \left.\frac{|F|^2}{|G|^2}\right|_{A=0} = |S_{22}|^2$ $T_r = \left.\frac{|B|^2}{|G|^2}\right|_{A=0} = |S_{12}|^2$

* relation between M and S:

$$\begin{pmatrix} F \\ G \end{pmatrix} = M\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \qquad \begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} A \\ G \end{pmatrix} = S\begin{pmatrix} A \\ G \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = M^{-1}\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix} \qquad \begin{pmatrix} A \\ G \end{pmatrix} = \begin{pmatrix} \bar e & \bar g \\ \bar f & \bar h \end{pmatrix}\begin{pmatrix} B \\ F \end{pmatrix} = S^{-1}\begin{pmatrix} B \\ F \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} e & g \\ f & h \end{pmatrix}^{-1} = \frac{1}{eh-fg}\begin{pmatrix} h & -g \\ -f & e \end{pmatrix} = \begin{pmatrix} \bar e & \bar g \\ \bar f & \bar h \end{pmatrix}$$

$$G = cA + dB \qquad B = \frac{1}{d}(-cA + G) = eA + gG$$

$$A = \bar a F + \bar b G \qquad F = \frac{1}{\bar a}(A - \bar b G) = fA + gH$$

$S = \Big(-c/d \quad 1/d \ \ldots \Big) = \Big(-c/d \quad 1/d \ \ldots \Big) = \frac{1}{d}\Big(-c \quad 1 \ \ldots \Big)$ (components

$\begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{a} \end{pmatrix} \quad a \begin{pmatrix} ad-bc & b \end{pmatrix}$ (of M)

$$A = \bar{e}\, B + \bar{g}\, F \qquad F = \frac{1}{\bar{g}}\left(A - \bar{e}\, B \right) = a\, A + b\, B$$

$$B = e\, A + g\, G \qquad G = \frac{1}{g}\left(-e A + B \right) = c\, A + d\, B$$

$$M = \begin{pmatrix} \frac{1}{g} & -\frac{\bar{e}}{\bar{g}} \\ -\frac{e}{g} & \frac{1}{g} \end{pmatrix} = \begin{pmatrix} \frac{eh-fg}{-g} & \frac{-h}{-g} \\ -\frac{e}{g} & \frac{1}{g} \end{pmatrix} = \frac{1}{g}\begin{pmatrix} fg-eh & h \\ -e & 1 \end{pmatrix}$$ (components of S)