Exam 1

Friday, October 30, 2015 03:40

University of Kentucky, Physics 520
Exam 1, 2015-10-30

Instructions: This exam is closed book and timed (50 minutes). Show intermediate work for partial credit. The last page is due Monday Nov. 2 at 11:00 AM sharp, and is also closed book. You may not consult any person or reference material besides your formula sheet. [100 pts maximum]

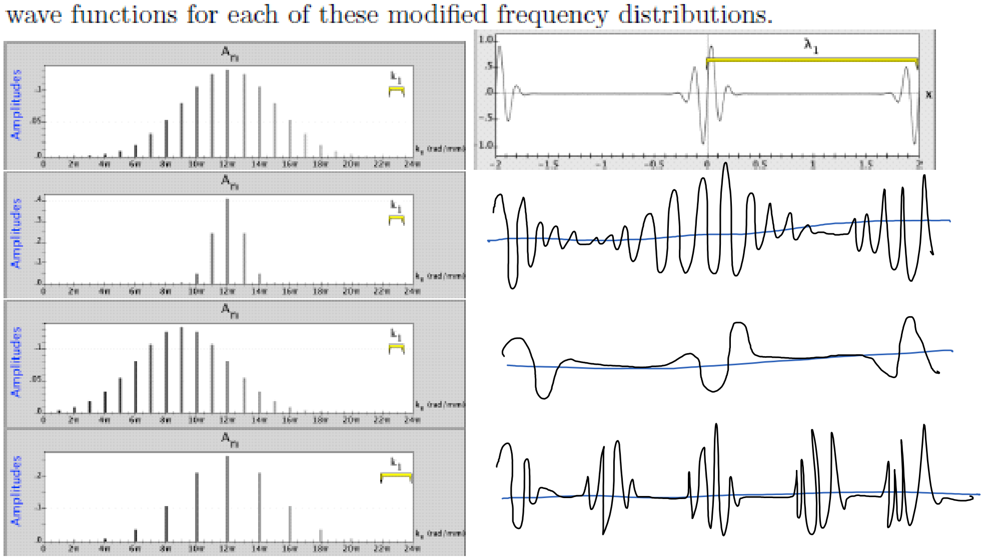
[5 pts] 1. Describe four roles played by light in the discovery of quantum mechanics. What relevant properties helped illuminate quantum mechanics [bring it out of darkness]?

- Planck described black body radiation (light)
- Einstein quantized light to describe the photoelectric effect.
- Compton show photons obeyed particle kinematics.
- Bohr showed transition frequency depended on ΔE
- light is massless (inherently quantum) and has broad frequency range.

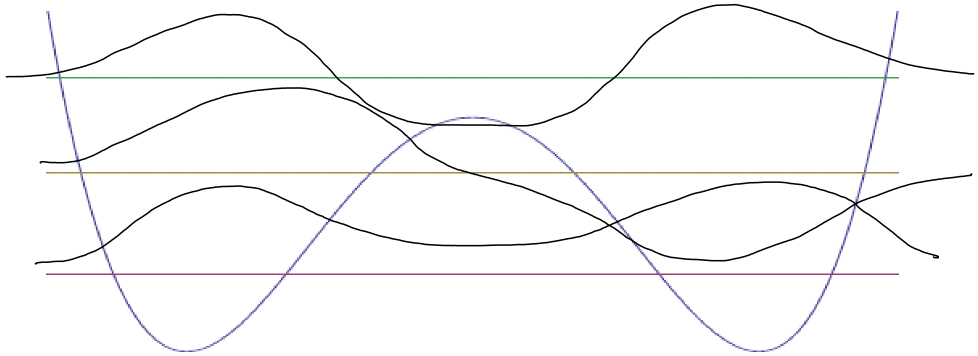
[5 pts] 2. When does matter behave like a wave and when does it act like a particle? What features of the wavefunction $\Psi(x, t)$ allow it to describe these two complementary properties?

- matter propagates unobserved as a wave
- matter interacts (is detected) as a particle.
- it is a continuous field that obeys a wave equation.
- it is a collection of probability amplitudes of observation

[5 pts] 3. Given the following wave function $\psi(x)$ with frequency component amplitudes A_k, draw wave functions for each of these modified frequency distributions.



[5 pts] 4. Sketch the three lowest energy wavefunctions of the Mexican hat potential:



[10 pts] 5. Given the initial wave function $\Psi(x,0) = 1/\sqrt{a}$ in an infinite square well on $0 < x < a$, calculate $\Psi(x,t)$. Note: $\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$ and $E_n = \frac{h^2 n^2}{8ma^2}$. Leave integrals unevaluated.

$$C_n = \langle \psi_n | \Psi_0 \rangle = \int_0^a \sqrt{\tfrac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\cdot\frac{1}{\sqrt{a}} = \frac{\sqrt{2}}{a}\cdot\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\Big|_0^a = \frac{\sqrt{8}}{n\pi}\,\delta_{n\ \mathrm{odd}}$$

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{i\omega_n t} = \sum_{n\ \mathrm{odd}} \frac{4}{n\pi\sqrt{a}}\sin\left(\frac{n\pi x}{a}\right) e^{i\frac{\pi h n^2 t}{4ma^2}}$$

[5 pts] 6. Calculate $\langle E\rangle$ for the wavefunction $\psi(x) = \sqrt{\frac{2}{3}}\,\psi_1(x) + \sqrt{\frac{1}{3}}\,\psi_2(x)$, where $\psi_n(x)$ are the normalized energy eigenstates $\mathcal{H}\psi_n(x) = E_n\psi_n(x)$. What are the possible outcomes of an energy measurement, the probability of each outcome, and the resulting wavefunction after measurement? What would change if the measurement were delayed by a time Δt?

$$\langle E\rangle = \sum_n |c_n|^2 E_n = \frac{2}{3}E_1 + \frac{1}{3}E_2$$

outcomes: a) $E = E_1$ $\quad P = \frac{2}{3}$ $\quad \psi \to \psi_1(x)$

b) $E = E_2$ $\quad P = 1/3$ $\quad \psi \to \psi_2(x)$

[5 pts] 7. Show that the wave function $\psi(x) = 1 - x^2$ is the unnormalized ground state of the potential $V(x) = \hbar^2 x^2/m(1-x^2)$ and calculate the energy of this state. on $-1 < x < 1$

$$\mathcal{H}\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}(1-x^2) + \frac{-\hbar^2 x^2}{m(1-x^2)}(1-x^2) = \frac{-\hbar^2}{2m}(-2) - \frac{\hbar^2 x^2}{m} = \frac{\hbar^2}{m}(1-x^2)$$

so $E_1 = \hbar^2/m$, ground state because no nodes.

[10 pts] 8. Quantize a hydrogen atom with circular orbits $F = \frac{Ze^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r}$ and $E = -\frac{1}{2}\frac{Ze^2}{4\pi\epsilon_0 r}$ to show that the n^{th} orbital has n de Broglie wavelengths around its circumference, and derive the Rydberg constant R_∞, where $\frac{1}{\lambda} = R_\infty(\frac{1}{n_f^2} - \frac{1}{n_i^2})$. for atomic spectra.

$$\frac{Ze^2}{4\pi\epsilon_0} = mv^2 r = Lv = n\hbar\cdot v \qquad v_n = \frac{Ze^2}{4\pi\epsilon_0 \hbar n}$$

$$r = \frac{L}{mv} = \frac{n\hbar}{mv} \qquad r_n = \frac{4\pi\varepsilon_0 \hbar^2 n^2}{m Z e^2}$$

$$E = -\frac{1}{2}\frac{Ze^2}{4\pi\varepsilon_0 r} = \frac{m Z^2 e^4}{2(4\pi\varepsilon_0)^2 \hbar^2 n^2}$$

$$L = rp = r\cdot\frac{h}{\lambda} = n\hbar \qquad 2\pi r = n\lambda \quad \text{[circumference]}$$

$R_\infty = 10973731.6/m$

$$\Delta E = hf = \frac{hc}{\lambda} \qquad \frac{1}{\lambda} = \frac{\Delta E}{hc} = \boxed{\frac{m Z^2 e^4}{8\pi^2\varepsilon_0^2 h^3 c}}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

Take-home portion:

9. The harmonic oscillator has the potential $V(x) = \frac{1}{2}m\omega^2 x^2$, where ω is the natural frequency. These integrals may be helpful: $I_0 = \sqrt{\frac{\pi}{\alpha}}$, $I_1 = \frac{1}{\alpha}$, $I_2 = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$, where $I_n = 2\int_0^\infty dx\, x^n e^{-\alpha x^2}$.

[5 pts] a) Calculate the ground state wave function $\psi_0(x)$.

$$a_- \psi_0(x) = 0 \quad \text{where} \quad a_- = \frac{1}{\sqrt{2\hbar\omega m}}(+i\hat{p} + m\omega\hat{x}) \quad \text{\& } \hat{p} = -i\hbar\frac{d}{dx}$$

$$\text{let } \xi = \sqrt{\frac{m\omega}{\hbar}}\,x \quad \text{then} \quad a_- = \frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right)$$

$$\left(\frac{d}{d\xi} + \xi\right)\psi_0(\xi) = 0 \qquad \frac{d\psi_0}{\psi_0} = -\xi\, d\xi \qquad \psi_0(\xi) = N e^{-\frac{1}{2}\xi^2}$$

$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = \sqrt{\frac{\hbar}{m\omega}}\, N^2 \int_{-\infty}^{\infty} d\xi\, e^{-\xi^2} = \sqrt{\frac{\hbar}{m\omega}}\, N^2 \sqrt{\pi} \qquad N = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

[10 pts] b) Calculate the uncertainty in position Δx of the ground state.

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\, x\,|\psi_0(x)|^2 = 0 \quad \text{odd integral over symmetric range.}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\, x^2 |\psi_0(x)|^2 = \sqrt{\frac{\hbar}{m\omega}}\, N^2 \int_{-\infty}^{\infty} d\xi \left(\frac{\hbar}{m\omega}\xi^2\right) e^{-\xi^2/2}$$

$$= \frac{1}{\sqrt{\pi}}\frac{\hbar}{m\omega}\cdot\frac{1}{2}\sqrt{\pi} = \frac{\hbar}{2m\omega}$$

thus $\Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2} = \sqrt{\frac{\hbar}{2m\omega}}$

[10 pts] c) Calculate the uncertainty in momentum Δp of the ground state using the operator $\hat{p} = -i\hbar\frac{d}{dx}$ in position space.

$$\langle p\rangle = \langle \psi_0 | -i\hbar \tfrac{d}{dx} | \psi_0\rangle = \sqrt{\frac{\hbar}{m\omega}}\, N^2 \int_{-\infty}^{\infty} d\xi\, e^{-\xi^2/2}\left(-i\hbar\sqrt{\frac{m\omega}{\hbar}}\underbrace{\frac{d}{d\xi}\right) e^{-\xi^2/2}}_{-\xi e^{-\xi^2/2}}$$

$= 0$ odd integrand.

$$\langle p^2\rangle = \sqrt{\frac{\hbar}{m\omega}}\, N^2 \int_{-\infty}^{\infty} d\xi\, e^{-\xi^2/2}\left(-\hbar^2\frac{m\omega}{\hbar}\frac{d^2}{d\xi^2}\right)e^{-\xi^2/2}$$

$$\frac{d^2}{d\xi^2} e^{-\xi^2/2} = \frac{d}{d\xi} -\xi e^{-\xi^2/2} = (-1+\xi^2)e^{-\xi^2/2}$$

$$= \frac{1}{\sqrt{\pi}}\hbar m\omega \int_{-\infty}^{\infty} d\xi\,(1-\xi^2)e^{-\xi^2/2} = \frac{1}{\sqrt{\pi}}\hbar m\omega\left(\sqrt{\pi} - \tfrac{1}{2}\sqrt{\pi}\right) = \tfrac{1}{2}\hbar m\omega$$

$$\Delta p = \left(\langle p^2\rangle - \langle p\rangle^2\right) = \sqrt{\frac{\hbar m\omega}{2}} \qquad \Delta x\cdot\Delta p = \sqrt{\frac{\hbar}{2m\omega}}\cdot\sqrt{\frac{\hbar m\omega}{2}} = \hbar/2$$

[10 pts] d) Calculate the ground state wave function $\phi_0(p)$ in momentum space.

$$\phi_0(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dx\, e^{ipx/\hbar}\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\sqrt{\frac{\hbar}{m\omega}}\, d\xi\, e^{i\beta\xi}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2} \qquad \beta = \frac{1}{\sqrt{\hbar m\omega}}\,p$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\pi m\omega\hbar}\right)^{1/4}\int_{-\infty}^{\infty} d\xi\, e^{-\frac{1}{2}\underbrace{(\xi^2 - 2i\beta\xi)}_{(\xi-i\beta)^2+\beta^2}} \qquad \text{let } \eta = \xi - i\beta$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\pi m\omega\hbar}\right)^{1/4} e^{-\beta^2/2}\underbrace{\int_{-\infty}^{\infty} d\eta\, e^{-\frac{1}{2}\eta^2}}_{\sqrt{\pi/\frac{1}{2}}} = \left(\frac{1}{\pi\hbar m\omega}\right)^{1/4} e^{-\frac{p^2}{2\hbar m\omega}}$$

[5 pts] e) Recalculate Δp in momentum space [using the wave function $\phi_0(p)$].

Check normalization: $\langle\phi|\phi\rangle = \int_{-\infty}^{\infty} dp\,|\phi(p)|^2 \qquad d\beta = \frac{1}{\sqrt{\hbar m\omega}}\,dp$

$$= \int_{-\infty}^{\infty}\sqrt{\hbar m\omega}\, d\beta\left(\frac{1}{\pi\hbar m\omega}\right)^{1/2} e^{-\beta^2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\beta\, e^{-\beta^2} = 1$$

$$\langle p \rangle = \int_{-\infty}^{\infty} dp\, p \cdot |\phi(p)|^2 = \int_{-\infty}^{\infty} dp\, p \cdot \frac{1}{\sqrt{\pi \hbar m\omega}} e^{-\frac{p^2}{\hbar m\omega}} = 0 \quad \text{(symmetry)}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} dp\, p^2 |\phi(p)|^2 = \frac{1}{\sqrt{\pi}} (\hbar m\omega) \underbrace{\int_{-\infty}^{\infty} d\beta\, \beta^2 e^{-\beta^2}}_{\sqrt{\pi}/2} = \frac{1}{2}\hbar m\omega \quad \text{same as c)}$$

[10 pts] f) Recalculate Δx and Δp using the ladder operators $\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle$, where $\hat{a}_\pm = \frac{1}{\sqrt{2\hbar\omega m}}(\mp i\hat{p} + m\omega\hat{x})$.

$$\hat{a}_\pm = \frac{\mp i\hat{p}}{\sqrt{2\hbar\omega m}} + \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} \qquad a_- = \begin{pmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ & & 0 & \sqrt{3} \\ & & & \ddots \end{pmatrix} \qquad a_+ = \begin{pmatrix} 0 & & & \\ \sqrt{1} & 0 & & \\ & \sqrt{2} & 0 & \\ & & \sqrt{3} & \ddots \end{pmatrix}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-) \qquad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & & \\ \sqrt{1} & 0 & \sqrt{2} & \\ & \sqrt{2} & 0 & \sqrt{3} \\ & & \sqrt{3} & \ddots \end{pmatrix} \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}} \begin{pmatrix} 0 & -\sqrt{1} & & \\ \sqrt{1} & 0 & \sqrt{2} & \\ & \sqrt{2} & 0 & \sqrt{3} \\ & & \sqrt{3} & \ddots \end{pmatrix}$$

$$\langle x \rangle = \langle 0|\hat{x}|0\rangle = \hat{x}_{00} = 0$$
$$\langle p \rangle = \langle 0|\hat{p}|0\rangle = \hat{p}_{00} = 0$$
$$\langle x^2 \rangle = \langle 0|\hat{x}^2|0\rangle = \hbar/2m\omega \qquad \hat{x}^2 = \frac{\hbar}{2m\omega}\begin{pmatrix} \sqrt{1\cdot 1} & 0 & \\ 0 & 1+2 & \\ & & \ddots \end{pmatrix} \qquad \hat{p}^2 = +\frac{\hbar m\omega}{2}\begin{pmatrix} \sqrt{1\cdot 1} & 0 & \\ 0 & 1+2 & \\ & & \ddots \end{pmatrix}$$
$$\langle p^2 \rangle = \langle 0|\hat{p}^2|0\rangle = \frac{\hbar m\omega}{2}$$

same as b), c).

Alternate: $\langle 0|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\, \underbrace{\langle 0|a_+}_{0} + \underbrace{a_-|0\rangle}_{0} = 0$

$$\langle 0|\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega}\langle 0|(a_+ + a_-)^2|0\rangle = \frac{\hbar}{2m\omega}\langle 0|a_+^2 + a_+a_- + a_-a_+ + a_-^2|0\rangle$$

$$= \frac{\hbar}{2m\omega}\left(0 + 0 + \underbrace{\langle 0|a_-}_{\langle 1|\sqrt{1}}\underbrace{a_+|0\rangle}_{\sqrt{1}|1\rangle} + 0\right) = \frac{\hbar}{2m\omega}$$

$$\langle 0|\hat{p}^2|0\rangle = \frac{-\hbar m\omega}{2}\langle 0|(a_+ - a_-)^2|0\rangle = \frac{-\hbar m\omega}{2}\left(0 + 0 - \langle 0|a_-a_+|0\rangle + 0\right) = \frac{\hbar m\omega}{2}$$