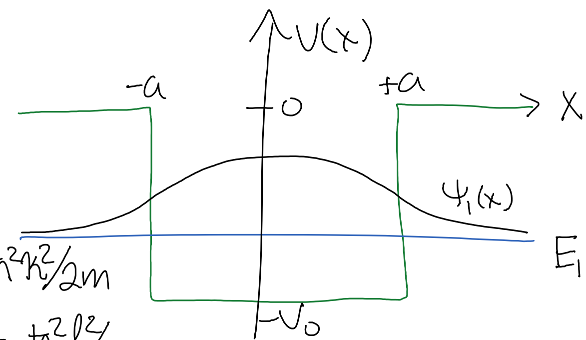
Exam 2 Solution

Monday, November 30, 2015 14:30

University of Kentucky, Physics 520
Exam 2, 2015-11-30

Instructions: This exam is closed book. Show intermediate work for partial credit. You may not consult any person or reference material besides your formula sheet. [~~100~~ 60 pts maximum]

[20 pts] 1. Calculate the energy of the ground state of a finite square well of width $2a$ and depth $V_0 = \hbar^2/2ma^2$, ie. $V(x) = -V_0$ if $|x| < a$ and 0 otherwise. Leave your answer as the graphical intersection of two curves, indicating the approximate value.



$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V\psi = E\psi$$

8 $x<a$: $\psi_I = Ae^{\kappa x} + Be^{-\kappa x}$ $\quad E = -\hbar^2\kappa^2/2m$

$|x|<a$: $\psi_{II} = C\cos(kl) + D\sin(kl)$ $\quad E + V_0 = \hbar^2 l^2/2m$

$x>a$: $\psi_{III} = Fe^{-\kappa x} + Ge^{\kappa x}$

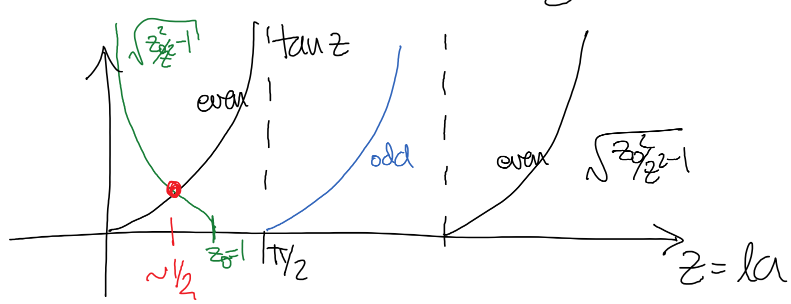
symmetry: $\psi(x) = \psi(-x)$ ground state $D=0$

8 B.C.: $\psi_{III}(\infty) = 0 \rightarrow G = 0$

$\psi_{II}(a) = \psi_{III}(a)$: $C\cos(la) = Fe^{-\kappa a}$

$\psi_{II}'(a) = \psi_{III}'(a)$: $-l\cdot C\cdot\sin(la) = -\kappa Fe^{-\kappa a}$

4 $\tan(la) = \kappa/l = \sqrt{\frac{z_0^2 - z^2}{z^2}} = \sqrt{z_0^2/z^2 - 1}$



$$-\frac{\hbar^2\kappa^2}{2m} + V_0 = \frac{\hbar^2 l^2}{2m}$$

$$(\kappa a)^2 = \underbrace{\frac{2mV_0a^2}{\hbar^2}}_{z_0^2=1} - \underbrace{(la)^2}_{z^2}$$

$$E_1 = -V_0 + \frac{\hbar^2 l^2}{2m} \approx -V_0 + \frac{\hbar^2}{8ma^2}$$

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[20 pts] 2. Calculate the probability that a quantum particle of mass m and energy $E > V_0$ incident from the left ($x < 0$) will bounce back from a step potential barrier of height V_0, i.e. $V(x) = 0$ if $x < 0$ and V_0 if $x > 0$.

$$\psi_I = Ae^{ikx} + Be^{-ikx} \qquad \frac{\hbar^2 k^2}{2m} = E$$

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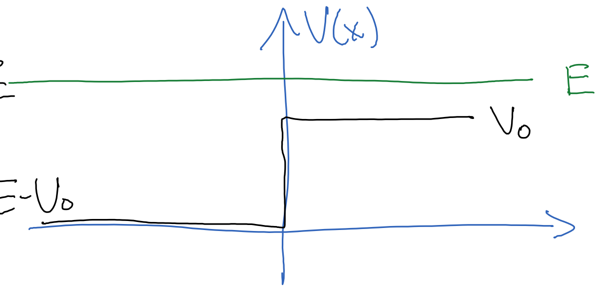
$$\psi_{II} = Fe^{ilx} + \cancel{G e^{-ilx}} \qquad \frac{\hbar^2 l^2}{2m} = E - V_0$$



$$\psi_I(0) = \psi_{II}(0): \qquad A + B = F \qquad (1)$$

$$\psi_I'(0) = \psi_{II}'(0): \qquad ikA - ikB = ilF \qquad (2)$$

8

$$ik(1) + (2): \qquad 2ikA = i(k+l)F \qquad F = \frac{2k}{k+l}A$$

$$il(1) - (2): \qquad i(l-k)A + i(l+k)B = 0 \qquad B = \frac{l-k}{l+k}A$$

prob. of bounce-back

4

$$R = \left|B/A\right|^2 = \left|\frac{l-k}{l+k}\right|^2 = \frac{\sqrt{E-V_0} - \sqrt{E}}{\sqrt{E-V_0} + \sqrt{E}}$$

[20 pts] 3. Given a two-state system with the Hamiltonian $\hat{H} = \begin{pmatrix} 0 & -B \\ -B & 0 \end{pmatrix}$ in the initial state $|\Psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the operator $\hat{X} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, calculate:

a) the time evolution of the state: $|\Psi(t)\rangle$

b) the time-dependent expectation value $\langle \hat{X} \rangle(t)$

c) the probability of measuring $X = 2$ and the probability of measuring $X = -2$ at time t.

a) solve for stationary states:

$$\hat{H}|\psi\rangle = E|\psi\rangle \qquad \begin{pmatrix} 0 & -B \\ -B & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{vmatrix} -E & -B \\ -B & -E \end{vmatrix} = E^2 - B^2 = 0 \qquad \begin{aligned} E_I &= -B \\ E_{II} &= +B \end{aligned}$$

eigenstate I: $\begin{pmatrix} B & -B \\ -B & B \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \psi_1 = \psi_2$

4 $\quad |I\rangle = n\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \langle I|I\rangle = |n|^2 \cdot 2 = 1 \qquad n = \frac{1}{\sqrt{2}}$

eigenstate II: $\begin{pmatrix} -B & -B \\ -B & -B \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \psi_1 = -\psi_2$

$|II\rangle = n\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \langle II|II\rangle = |n|^2 \cdot 2 = 1 \qquad n = \frac{1}{\sqrt{2}}$

thus $\quad |I\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \qquad |II\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$

$$|\Psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_I|I\rangle + c_2|II\rangle$$

$$c_I = \langle I|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

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$$c_{II} = \langle II|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(1\ -1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-1}{\sqrt{2}}$$

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}|I\rangle e^{iBt/\hbar} - \frac{1}{\sqrt{2}}|II\rangle e^{-iBt/\hbar}$$

b) $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\right)e^{iBt/\hbar} - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)\right)e^{-iBt/\hbar}$

$$= \tfrac{1}{2}\left(e^{iBt/\hbar} - e^{-iBt/\hbar}\right)|1\rangle + \tfrac{1}{2}\left(e^{iBt/\hbar} + e^{-iBt/\hbar}\right)|2\rangle$$

4

$$= \underbrace{i\sin(Bt/\hbar)}_{c_1}|1\rangle + \underbrace{\cos(Bt/\hbar)}_{c_2}|2\rangle = \begin{pmatrix} i\sin Bt/\hbar \\ \cos Bt/\hbar \end{pmatrix}$$

$$\langle X\rangle = P_1(t)\cdot x_1 + P_2(t)\cdot x_2$$

4

$$= \sin^2(Bt/\hbar) \cdot 2 + \cos^2(Bt/\hbar) \cdot 1$$

$$= 2 - \cos^2(Bt/\hbar) = 1 + \sin^2(Bt/\hbar)$$

$$= 3/2 - 1/2 \cos(2Bt/\hbar)$$

4

c) $P(x=2)(t) = |c_1|^2 = \sin^2(Bt/\hbar)$

$P(x=-2)(t) = 0$ not an eigenstate of $\hat{X}$