University of Kentucky, Physics 520
Homework #9, Rev. A, due Friday, 2016-11-04

0. Griffiths [2ed] App. A #4, #9, #10.

1. The **complex plane** $\{\boldsymbol{w} = (x, y)\}$ is the vector space of *real* and *imaginary* components of $w = x + iy \in \mathbb{C}$, with the additional operation of multiplication, similar to the *general linear group* GL(n) of $n \times n$ matrices. We explore this analogy by generalizing the exponential map $e^{i\phi}$.

a) Identify the two basis elements of the complex plane in the above representation of w.

b) Show that the dot and cross product of two points $\boldsymbol{w}_1 = (x_1, y_1)$ and $\boldsymbol{w}_2 = (x_2, y_2)$ are given by the real and imaginary parts of the complex product $w_1^* w_2 = \boldsymbol{w}_1 \cdot \boldsymbol{w}_2 + i(\boldsymbol{w}_1 \times \boldsymbol{w}_2)_z$, where $w^* = x - iy$ is the *complex conjugate* of w. Identify the symmetric and antisymmetric terms of this product. Thus the complex product $|w|^2 = w^* w$ equals the vector product $\boldsymbol{w} \cdot \boldsymbol{w}$.

c) Show graphically that the operator $w \to iw$ rotates the point w 90° CCW about the origin.

d) Show graphically that the operator $1 + i\,d\phi : w \mapsto w + iw\,d\phi$ preserves the magnitude of w (assuming $d\phi^2 = 0$), but rotates it CCW by the infinitesimal angle $d\phi$.

e) Obtain a finite rotation from an infinite number of $d\phi$ rotations as follows: formally integrate the equation $dw = iw\,d\phi$ with the initial condition $w|_{\phi=0} = w_0$ to obtain the rotation formula $w(\phi) = R_\phi w_0$, where $R_\phi = e^{i\phi}$. Use this result to justify the identity $e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$.

f) Separate the Taylor expansion of $e^{i\phi}$ into $x+iy$ to prove Euler's formula, $e^{i\phi} = \cos\phi + i \sin\phi$.

g) Show that complex multiplication by i is equivalent to the vector operator $\hat{\boldsymbol{z}}\times$.

h) Determine the matrix representation M_z of the operator $\hat{\boldsymbol{z}}\times$, where $M_z \boldsymbol{r} = \hat{\boldsymbol{z}} \times \boldsymbol{r}$. Do the same for M_x and M_y to show that $\boldsymbol{v}\times = \boldsymbol{v} \cdot \boldsymbol{M} = v_x M_x + v_y M_y + v_z M_z$ is the matrix representation of $\boldsymbol{v}\times$ for any vector $\boldsymbol{v}$. *Hint:* You should find that the vector of matrices $\boldsymbol{M} \sim (M_i)_{jk} = \varepsilon_{ijk}$ (cross product tensor) is completely antisymmetric in indices i, j, k.

i) Restricting to the xy-plane, show that the 2×2 matrix $M_z^2 = -I$ analogous to $i^2 = -1$, and the matrix for a CCW rotation ϕ is $R_\phi = e^{M_z \phi} = I \cos\phi + M_z \sin\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$. *Hint:* the *exponential* of a matrix M_z is defined by its Taylor expansion as in part f). Note that in general, the matrix $R_{\boldsymbol{v}}$ for a CCW rotation by angle $v = |\boldsymbol{v}|$ about the $\hat{\boldsymbol{v}}$-axis can be written as $R_{\boldsymbol{v}} = I \cos v + \boldsymbol{M}\cdot\hat{\boldsymbol{v}} \sin v + \hat{\boldsymbol{v}}\hat{\boldsymbol{v}}^T(1 - \cos v)$, where the third term corrects for the non-rotating projection along $\boldsymbol{v}$.

j) Calculate the eigenvalues and eigenvectors of $M_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to show that $M_z = VWV^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $e^{M_z\phi} = Ve^{W\phi}V^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Multiply this out to verify part i). Thus real Hermitian matrices have real eigenvalues while antiHermitian matrices have Tr=0 and imaginary eigenvalues. The exponential of a Hermitian matrix is positive definite with real positive eigenvalues, while the exponential of an antiHermitian matrix is unitary with Det=1 and unit modulus eigenvalues. This is the normal matrix analogy: Hermitian/antiHermitian matrices are analogous to the real/imaginary numbers.