L05-Anatomy of a Wave

Sunday, September 4, 2016 09:43

* We will be studying "wave mechanics", Schrödinger's program of quantizing energy states in terms of standing waves.
- Before studying quantum wave functions, we should understand the features of classical waves.
- This will lead to a natural motivation of the T.D.S.E.

* What is a wave?

- A wave is a collective mode of oscillation that can transfer energy and momentum from one location to another without the corresponding transfer of material (mass)

- It is characterized by properties of the underlying medium:
  tension, density $\rightarrow$ velocity (dispersion relation), impedance (energy storage)
  And also by properties of the mode of oscillation:
  wavelength, frequency (spectrum), amplitude, polarization
  Compare: particles have static moments (mass, centre of mass, inertia) and degrees of freedom: position $\vec{x}(t) \rightarrow$ velocity $\rightarrow$ energy, momentum

- Energy is transferred as it oscillates between different forms in the medium, for example potential [tension] & kinetic [inertia].
  Compare: particles transfer pure kinetic energy by motion

- The state is described by a wave function, which evolves according to a PDE wave equation
  Compare: particles have a trajectory following an ODE equation of motion

- Mechanical waves are composed of particles (usually bound)
  0-d {single particle: oscillator $m\ddot{x}+b\dot{x}+kx=f(t) \rightarrow x = A\cos(\omega t+\phi)$
  {multiple particles: collective mode: $M\mathbb{x} + K\mathbb{x} = 0 \rightarrow \mathbb{x} = \sum_i \eta_i \cos(\omega_i t + \phi_i)$
  1,2,3-d {continuous medium $x_i \rightarrow f(\vec{x})$ "$\vec{x}$" is now a particle index!
  Other waves [E&M, quantum, gravitational] travel through a medium of underlying fields, not particles.

* What is the difference between a classical particle and wave?

PROPERTY	PARTICLE	WAVE
material properties:	mass "m", inertia "I"	velocity "v", impedance "Z"

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material properties:	mass "m", inertia "I"	velocity "v", impedance "Z"
degrees of freedom:	$\vec{r}(t)$ trajectory (vector)	$\Phi(\vec{r},t)$ wave function (field)
(Fourier space)		Amplitude$(\vec{k},\omega)$, polarization
dynamics:	$\vec{F}=m\vec{a}$ [ODE]	$(\nabla^2 + \frac{1}{v^2}\frac{\partial^2}{\partial t^2})\,\Phi(\vec{r},t)=0$ [PDE]
dispersion	$E = p^2/2m$	$v=\lambda\nu=\omega/k$
conservation	$(E,\vec{p})=$ const	$\nabla\cdot(\vec{S},\overleftrightarrow{T}) + \partial_t(u,\vec{g})=0$
quantization	mass	modes (standing waves)
sociality	individual	collective motion of particles
interactions	collision, forces	reflection/refraction/diffraction

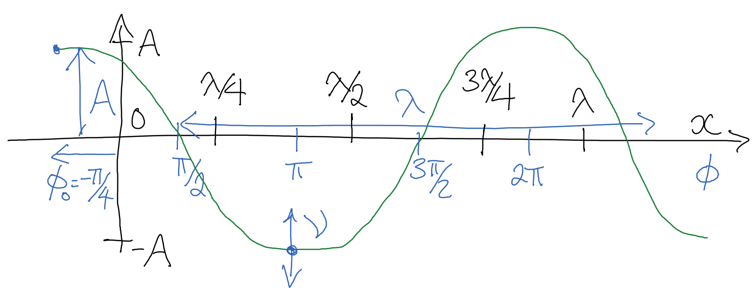
* Examples of waves:

1) Material waves, phase: solid, liquid, gas
   dimension: 1-d (tension), 2-d (surface tension, gravity), 3-d (body)
   Examples: string: tension T, linear mass density μ
   gravity: gravity g, (mass density cancels!)
   acoustic: pressure P, density ρ.
   seismic: stress/strain moduli, density.
2) Electromagnetic waves, 1,2,3-d wave guides.
   E, B fields, permittivity ε (tension), permeability μ (inertia)
3) Gravitational waves
   metric tensor field, energy/curvature G, ?
4) Quantum mechanical waves
   probability amplitude, kinematic dispersion $H=\frac{p^2}{2m}+V$

* Properties of a one-dimensional wave function

$$\Psi(x,t)=A_0\cos(\underbrace{kx-\omega t-\phi_0}_{\text{phase } \phi})$$

- It's best to convert everything to radians and think in terms of phase.

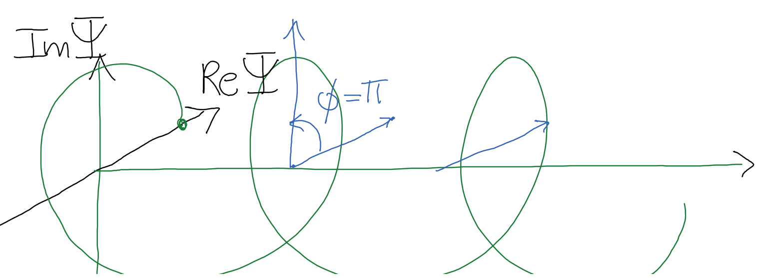


$\omega=2\pi\nu=\frac{2\pi}{T}$ $\quad k=\frac{2\pi}{\lambda}$ "wave number" $\quad v=\lambda\nu=\frac{\omega}{k}$ "dispersion"

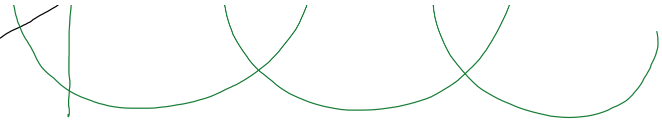
* Complex amplitudes

$e^{i\phi}=\cos\phi + i\sin\phi =$ "cis(ϕ)"

$\Psi(x,t)=\mathrm{Re}[A_0e^{-i\phi_0}e^{i(kx-\omega t)}]$



$$\Psi(x,t) = \mathrm{Re}\left[A_0 e^{-i\phi_0} e^{i(kx-\omega t)}\right]$$



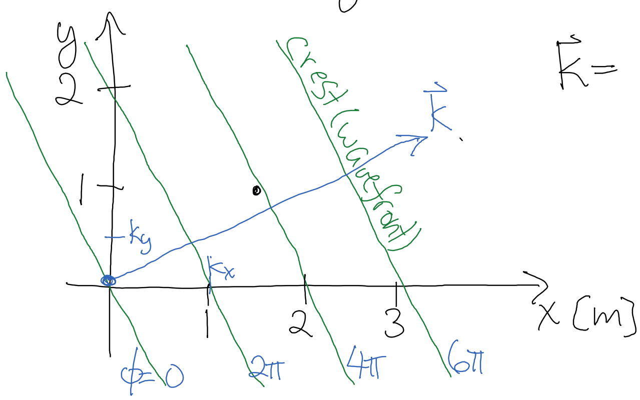
Advantage: everything factors! (separation of variables)

- complex amplitude includes phase
- Imaginary part captures the velocity (1st order equation)
- Real exponenent represents attenuation

Exponentials are "eigenfunctions" of derivative operators.

$$\underbrace{\boxed{\frac{\partial}{\partial x}}}_{\text{operator}} e^{ikx} = \underbrace{ik}_{\text{eigenvalue}} e^{ikx} \qquad \underbrace{\boxed{\frac{\partial}{\partial t}}}_{\text{operator}} e^{-i\omega t} = \underbrace{-i\omega}_{\text{eigenvalue}} \underbrace{e^{-i\omega t}}_{\text{eigenfunction}}$$

* Waves in higher dimension (2 or 3-d)



$$\vec{k} = (k_x, k_y) = (2,1)\cdot \pi \ \frac{\text{rad}}{\text{m}} \qquad \omega = 2\pi \ \frac{\text{rad}}{\text{s}}$$

$$\vec{k}\cdot\vec{r} = k_x x + k_y y = \phi = \text{const}$$

$$v_n = \frac{\omega}{\vec{k}\cdot\hat{n}} \qquad v_x = 1\ \text{m/s} \qquad v_y = 2\ \text{m/s}$$

$$\underbrace{\boxed{\nabla}}_{\text{operator}} e^{i\vec{k}\cdot\vec{r}} = \underbrace{i\vec{k}}_{\text{eigenvalue}} \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{eigenfunction}}$$

* Homogenous waves: pure frequency waves are independent of position (besides phase).

It is easy to come up with many differential equations:

$$\mathcal{O}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) A_0 e^{i(kx-\omega t)} = \mathcal{O}(ik_x, -i\omega)\, A_0 e^{i(kx-\omega t)} = 0$$

For example, for a wave on a string,

$$\left[T\frac{\partial^2}{\partial x^2} - \mu\frac{\partial^2}{\partial t^2}\right]\Psi(x,t) = 0 \quad \text{"wave equation"}$$

$$\left[T(-k^2) - \mu(-\omega^2)\right]\Psi(x,t) = 0$$

$$Tk^2 - \mu\omega^2 = 0 \quad \text{"dispersion relation"}$$

$$v_\phi = \omega/k = \sqrt{T/\mu}$$