L06-Wave equation and dispersion relations

Sunday, September 4, 2016 10:04

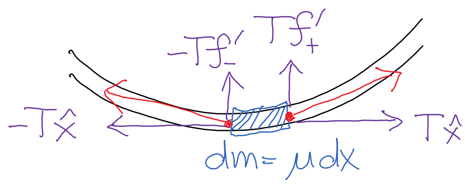
* We studied the wave function last time (kinematics) now lets look at its evolution (dynamics), which is governed by the wave equation.

Linear differential equations allow superposition of waves, which allows for constructive & destructive interference.

ex: "$\frac{\partial}{\partial x}$" is a linear operator: $\partial_x[\psi_1 + \psi_2] = \frac{\partial}{\partial x}\psi_1 + \frac{\partial}{\partial x}\psi_2$

Homogeneous differential equations are independent of position, and have pure frequency waves as solutions.

ex: wave on a string:
The balanced tension keeps the rope from moving left/right, but curvature ($\frac{\partial^2}{\partial x^2}$) causes acceleration → oscillation up/down



$dF = dm\, a$

$d(Tf') = \mu dx \cdot \ddot{f}$

$Tf'' = \mu \ddot{f}$

$T\frac{\partial^2}{\partial x^2} - \mu\frac{\partial^2}{\partial t^2} f(x,t) = 0$ let $f(x,t) = Ae^{i(kx-\omega t)}$

$[-Tk^2 + \mu\omega^2] Ae^{i(kx-\omega t)} = 0$

* $\mu\omega^2 = Tk^2$ is the dispersion relation $\omega(k)$

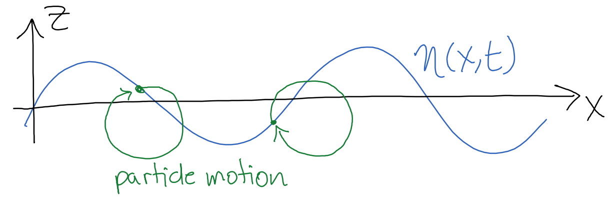
a) it determines the phase velocity $v_\phi = \omega/k$ for each k
b) it determines the group velocity $v_g = d\omega/dk$ of a pulse
c) it determines the dispersion $\beta = d^2\omega/dk$ (spreading out)

* In general, the homogenous wave equation is equivalent to its dispersion relation, since ∂_x and ∂_t can be replaced by ik and $i\omega$.

$\mathcal{O}(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}) A_0 e^{i(kx-\omega t)} = \mathcal{O}(ik_x, -i\omega) A_0 e^{i(kx-\omega t)} = 0$ [wave eq.]

$$\Rightarrow \quad \mathcal{O}(ik_x, -i\omega) = 0 \qquad \text{[dispersion relation]}$$

ex: gravity wave (water) ignoring surface tension, and infinitely deep water:



Let $\eta(x,t)$ be the height of the wave above equilibrium. and $\phi(x,z,t)$ be a flow potential of water at height z. It's gradient is the velocity of the water $\vec{v}(x,z,t) = \nabla\phi$. $\phi(x,z,t)$ can be formed if the flow is irrotational, ie. $\nabla \times \vec{v} = 0$. The water is incompressible: $\nabla \cdot \vec{v} = \nabla^2\phi = 0$

The solution to Laplace's equation $\nabla^2\phi = 0$

is $\phi = \frac{\omega A}{k} e^{-kz} \sin(kx - \omega t)$, which satisfies the

boundary conditions $\phi \to 0$ as $z \to -\infty$ (far from surface) and $v_z = \partial_z \phi = \partial_t \eta$ at $z=0$ (on the surface)

where $\eta(x,t) = A \cdot \cos(kx - \omega t)$

To get the dispersion relation, apply the Bernoulli equation:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = -\nabla PV + m\vec{g}$$

$$\int dz \left[\frac{\partial}{\partial t} \nabla\phi = -\frac{\nabla P}{\rho} + \vec{g} \right] \qquad \rho \equiv \frac{m}{V}$$

Divide by mass and integrate over height z.

$$\frac{\partial\phi}{\partial t} = \underbrace{g\eta}_{\text{gravity}} - \underbrace{\cancel{\frac{\gamma}{\rho}\frac{\partial^2}{\partial x^2}\eta}}_{\text{surface tension}}$$

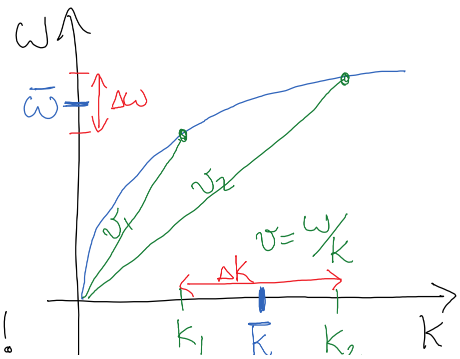
Ignore surface tension and substitute ϕ, η at the surface $z=0$

$$\left(\frac{\omega^2}{k} - g\right) A\cos(kx - \omega t) = 0$$

Dispersion relation: $\omega = \sqrt{gk}$

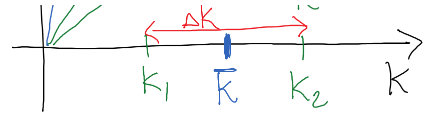
$$v_\phi = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

Tsunami: $\lambda = 10\,\text{km}$
$v_\phi = 450\,\text{km/hr}$!

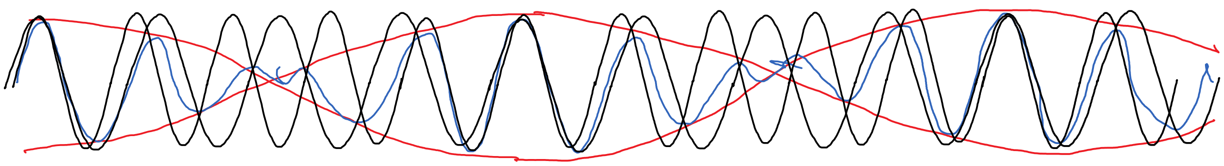


Different wave lengths have different velocities!

Different wavelengths have different velocities!



* What about the group velocity (of a wave packet)?



$$f(x,t) = A e^{i(k_1 x - \omega_1 t)} + A e^{i(k_2 x - \omega_2 t)}$$

let $\bar{k} = \frac{1}{2}(k_1 + k_2)$

$\Delta k = k_2 - k_1$ (bandwidth)

$$= A e^{i(\bar{k}x - \bar{\omega}t)} \left[e^{-i(\Delta k x - \Delta\omega t)} + e^{i(\Delta k x - \Delta \omega t)} \right]$$

$$= A \underbrace{e^{i(\bar{k}x - \bar{\omega}t)}}_{\text{phase/carrier wave}} \cdot 2 \underbrace{\cos(\Delta k \cdot x - \Delta\omega t)}_{\text{group/packet/beats/modulation.}}$$

$\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$

$\Delta\omega = \omega_2 - \omega_1$

$$v_\phi = \bar{\omega}/\bar{k} \qquad v_g = \Delta\omega/\Delta k \rightarrow d\omega/dk$$