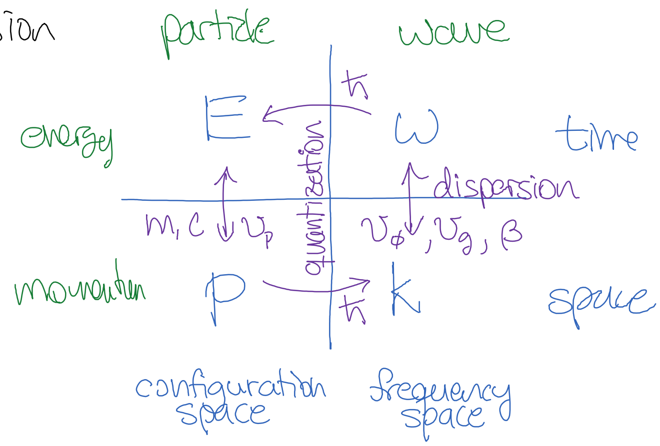
L07-Schrödinger Equation and Heisenberg Uncertainty

Friday, September 9, 2016 17:31

* Review: Quantization & Dispersion are the key ingredients of the Schrödinger Eq.



- wave / particle duality lead to quantization of both radiation & matter
- space/time is connected by dispersion relations, which are equivalent to the underlying wave equation, but also relate particle energy and momentum.

* Example: Dispersion relations in E&M:

Coulomb/Ampere /Faraday/Maxwell $\quad \nabla\cdot\vec{D} = \rho \quad \nabla\times\vec{E} = -\partial_t\vec{B} \quad \vec{D} = \varepsilon\vec{E}$ "stiffness"

$\nabla\cdot\vec{B} = 0 \quad \nabla\times\vec{H} = \vec{J} + \partial_t\vec{D} \quad \vec{B} = \mu\vec{H}$ "inertia"

We obtain the wave equation by substituting 2 curl eqs:

$$\nabla\nabla\cdot\vec{E} - \nabla^2\vec{E} = \nabla\times(\nabla\times\vec{E}) = -\partial_t\mu\nabla\times\vec{H} = -\partial_t\mu(\vec{J} + \partial_t\varepsilon\vec{E})$$

$$(-\mu\varepsilon\partial_t^2 + \nabla^2)\vec{E} = \underbrace{-\partial_t\mu\vec{J} - \nabla\rho/\varepsilon}_{\text{source terms}}$$ E&M wave equation

$$(\mu\varepsilon\omega^2 - k^2)\vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} = 0$$ Dispersion relation

a) $v = \omega/k = \boxed{\frac{1}{\sqrt{\varepsilon\mu}}} = \frac{c}{n}$ where $c \equiv \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ "speed of light"

$n(\lambda) = \sqrt{\varepsilon_r\mu_r} \approx 1 + A(1 + B/\lambda^2)$ "Cauchy's dispersion formula"

b) $U = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 H^2 = \varepsilon_0 E^2 = \mu_0 H^2$ "energy density"

$\vec{S} = \vec{E}\times\vec{H} = ZH^2\hat{k} = U\vec{v}$ $\quad \overbrace{E = HZ}^{V = IR}$ "Poynting vector"

$i\vec{k}\times\vec{E} = i\omega\mu\vec{H} \quad Z = E/H = \mu\,\omega/k = \mu v = \frac{1}{\varepsilon v} = \boxed{\sqrt{\mu/\varepsilon}}$ "Charateristic Impedance"

Units: $\frac{1}{4\pi\varepsilon_0} = 9\times10^9 \frac{Nm^2}{A^2s^2}$ $\quad \frac{\mu_0}{4\pi} \equiv 10^{-7} N/A^2$ $\quad c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8 m/s$

$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \mu_0 c = 30 \times 4\pi\,\Omega = 377\,\Omega$ $\quad P = ZI^2$ (the power radiated by an antenna.)

$E = pc$ (energy and momentum carried by the wave)

This is consistent with quantization: $\hbar(\omega = ck)$

* Dispersion of Quantum Mechanical matter waves:

$E^2 = (pc)^2 + (mc^2)^2$ (Einstein S.R.) $\Rightarrow$ $T = \frac{p^2}{2m} = \frac{1}{2}mv^2$ (N.R. Kinetic energy)

Matter has the extra property of mass (photons are massless)

$\underbrace{\hbar\omega}_{E} = \underbrace{\frac{\hbar^2k^2}{2m}}_{T} + \underbrace{\text{potential energy}}_{V}$ $\quad \partial_t = -i\omega$, $\nabla = i\vec{k}$ applied to $\Psi(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\underbrace{i\hbar\partial_t}_{\hat{E}}\Psi = \underbrace{\frac{-\hbar^2}{2m}\nabla^2}_{\hat{T}}\Psi + \underbrace{V(r)}_{\hat{V}}\Psi$ T.D.S.E. "Time Dependent Schrödinger Eq."

"operators."

* What is the velocity v_ϕ of a quantum wave?

$v_\phi = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{p}{2m} = \frac{1}{2}v_p$ half the speed of the particle!?

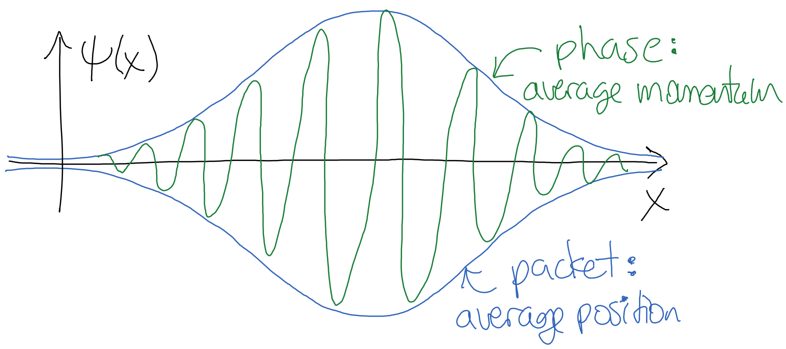
This goes back to what is a particle in Q.M.?
It is a "wave packet", otherwise the wave function goes on forever and is not normalizable.
The wave function should oscillate within some envelope, which determines "where" the particle is, in a probabilistic sense. (next class).

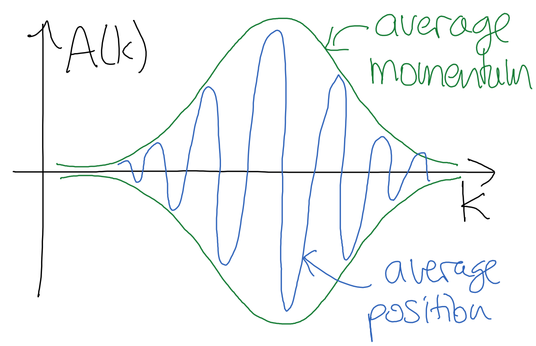
The group velocity is the speed of this envelope:

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(\frac{\hbar k^2}{2m}\right) = \frac{\hbar k}{m} = \frac{p}{m} = v_p$$

* A wave packet is smeared over both position AND momentum.
It is the superposition of a whole spectrum of pure frequency waves.

* A wave packet is smeared over both position AND momentum. It is the superposition of a whole spectrum of pure frequency waves.





$$\Psi(x) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$$

where

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ikx} \cdot \Psi(x)$$

"Fourier transformation" (or decomposition)

* A direct consequence of Fourier decomposition is the "Heisenberg Uncertainty Principle"

$$\Delta k \cdot \Delta x \geq 1/2 \quad \Rightarrow \quad \Delta p \cdot \Delta x \geq \hbar/2$$
$$\Delta \omega \cdot \Delta t \geq 1/2 \quad \Rightarrow \quad \Delta E \cdot \Delta t \geq \hbar/2$$

The position and momentum of a particle are both intertwined in the wave function and cannot be specified separately.
"Bohr's Complementarity Principle" (wave particle duality)

The minimum uncertainty wave $\Delta k \cdot \Delta x = \hbar/2$ is a Gaussian wave packet

$$\Psi(x) = \int dk\, A(k)\, e^{ikx} \qquad A(k) = N e^{\frac{1}{4}\left(\frac{k-k_0}{\Delta k}\right)^2}$$

Explore this using the PhET Fourier Decomposition Applet:

https://phet.colorado.edu/en/simulation/fourier