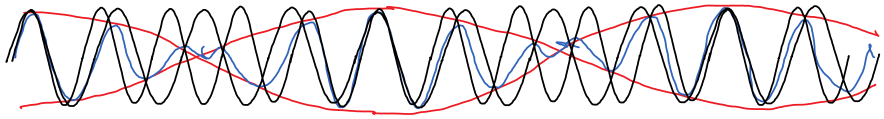
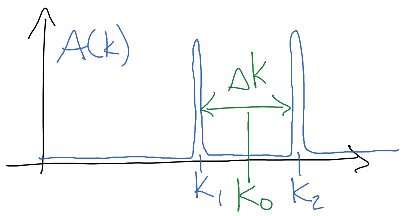
L08-Gaussian Wave Packets

Tuesday, September 13, 2016 17:41

* Review beats of two similar frequencies





Generalization: Fourier Series - synthesis of $f(x)$ with period $\lambda_1 = \frac{2\pi}{k_1}$

$$f(x) = \sum_n A_n \cos(k_n x) + B_n \sin(k_n x) \qquad \text{or} \qquad \sum_{n=-\infty}^{\infty} C_n e^{ik_n x} \qquad k_n = n \cdot k_1$$

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \frac{2}{\lambda_1}\int_0^{\lambda_1} f(x)\begin{bmatrix}\cos(k_x) \\ \sin(k_x)\end{bmatrix} dx \qquad C_n = \frac{1}{\lambda_1}\int_0^{\lambda} f(x)\, e^{-ik_n x}\, dx$$

Continuous limit: $k_1 \to dk$, $\sum_n A_n \to \int dk\, A(k)$ is the Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int dk\, A(k)\, e^{ikx} \quad \text{and it's inverse:} \quad A(k) = \frac{1}{\sqrt{2\pi}}\int dx\, f(x)\, e^{-ikx}$$

note: sometimes the $\frac{1}{\sqrt{2\pi}}$'s are combined $\tilde{f}(k) = \frac{1}{2\pi}\int dx\, f(x)\, e^{-ikx}$

* explore PhET applet https://phet.colorado.edu/en/simulation/fourier
  - $k_1 = \frac{2\pi}{\lambda_1}$ fundamental frequency, repetition period λ_1
    frequency of the wave packets [discrete only]
    $k_1 = dk \to 0$ for Fourier transform
  - A_n peaked at $k_0 = nk_1$ carrier (phase) frequency
    apparent frequency of the wave
  - Δk = frequency bandwidth (modulation)
    sharpness of packet in position space
    highest frequency of wave packet

* useful family of integrals: $\boxed{I_n \equiv \int_0^{\infty} dx\, x^n e^{-\alpha x^2}}$

$$I_1 = \int_0^{\infty} x\, e^{-\alpha x^2} dx = \frac{1}{2\alpha}\int_0^{\infty} e^{-u}\, du = \frac{1}{2\alpha} \qquad u = \alpha x^2,\ du = \alpha \cdot 2x\, dx$$

$$I_0^2 = \int_0^{\infty}\int_0^{\infty} dx\, dy\, e^{-\alpha(x^2+y^2)} = \int_0^{\infty} s\, ds\, e^{-\alpha s^2}\int_0^{\pi/2} d\phi = \frac{\pi}{2} I_1 = \frac{\pi}{4\alpha}$$

$$I_{n+2} = \int_0^{\infty} x^{n+2}\, dx\, e^{-\alpha x^2} = -\frac{\partial}{\partial\alpha} I_n = \frac{n+1}{2\alpha} I_n \qquad I_2 = -\frac{\partial}{\partial\alpha}\frac{1}{2}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

n	0	1	2	3	4	5	6
$\frac{1}{2}(n+1)$	1/2	1	3/2	2	5/2	3	7/2
$2I_n$	$\sqrt{\frac{\pi}{\alpha}}$	$\frac{1}{\alpha}$	$\frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$	$\frac{1}{\alpha^2}$	$\frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$	$\frac{2}{\alpha^3}$	$\frac{15}{8}\sqrt{\frac{\pi}{\alpha^7}}$

$$\Gamma(t) \equiv \int_0^\infty u^{t-1} e^{-u}\,du \qquad \text{let } u = \alpha x^2 \quad du = \alpha \cdot 2x\,dx$$

$$= \int_0^\infty (\alpha x^2)^{t-1} e^{-\alpha x^2} \cdot \alpha \cdot 2x\,dx$$

$$= 2\alpha^t \int_0^\infty x^{2t-1} e^{-\alpha x^2}\,dx$$

$$= 2\alpha^t I_{2t-1} \qquad \begin{matrix} n = 2t-1 \\ t = \frac{1}{2}(n+1) \end{matrix}$$

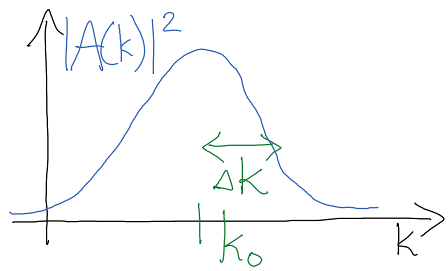
$$\text{so} \quad \boxed{2I_n = \frac{\Gamma(\frac{1}{2}(n+1))}{\alpha^{\frac{1}{2}(n+1)}}}$$

$$\Gamma(t+1) = \int_0^\infty u^t e^{-u}\,du$$
$$= \int_0^\infty u^t\,d(e^{-u})$$
$$= -\cancel{u^t e^{-u}\Big|_0^\infty} + \int_0^\infty e^{-u}\,du^t$$
$$= t\int_0^\infty u^{t-1} e^{-u}\,du$$

$$\left.\begin{matrix} \Gamma(t+1) = t\,\Gamma(t) \\ \Gamma(n+1) = n! \\ \Gamma(\frac{1}{2}) = \sqrt{\pi} \end{matrix}\right\} \text{properties of } \Gamma(t)$$

* Example: Gaussian wave packet (following Gasiorowicz 2-6)

$$A(k) = N e^{-\frac{1}{4}\left(\frac{k-k_0}{\Delta k}\right)^2} \qquad |A(k)|^2 \text{ normal distribution}$$



Normalization: $\int |\Psi(x)|^2 dx = \int |A(k)|^2 dk = 1$

$$\int_{-\infty}^{\infty} N^2 e^{-\frac{1}{2}\left(\frac{k-k_0}{\Delta k}\right)^2} dk \qquad \chi = \frac{k-k_0}{\Delta k} \quad dk = \Delta k\,d\chi$$

$$= N^2 \Delta k \int_{-\infty}^{\infty} e^{-\chi^2/2} d\chi = N^2 \Delta k \cdot \sqrt{2\pi} = 1 \quad \text{so} \quad N = \frac{1}{\sqrt{2\pi \Delta k}}$$

Mean: $\langle k \rangle \equiv \int_{-\infty}^{\infty} N^2 e^{-\frac{1}{2}\left(\frac{k-k_0}{\Delta k}\right)^2} k\,dk = N^2 \Delta k \int_{-\infty}^{\infty} e^{-\chi^2/2} (\overset{\text{odd}}{\cancel{\Delta k \chi}} + k_0)\,d\chi = k_0$

RMS: $\langle k^2 \rangle \equiv \int_{-\infty}^{\infty} N^2 e^{-\frac{1}{2}\left(\frac{k-k_0}{\Delta k}\right)^2} k^2 dk = N^2 \Delta k \int_{-\infty}^{\infty} e^{-\chi^2/2} \underbrace{(\Delta k \chi + k_0)^2}_{\Delta k^2 \chi^2 + \cancel{2\Delta k \chi k_0} + k_0^2} d\chi$

$$= N^2 \Delta k^3 \int_{-\infty}^{\infty} e^{-\chi^2/2} \chi^2 d\chi + k_0^2 = \Delta k^2 + \langle k \rangle^2 \quad \text{so} \quad \sigma_k = \Delta k$$

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