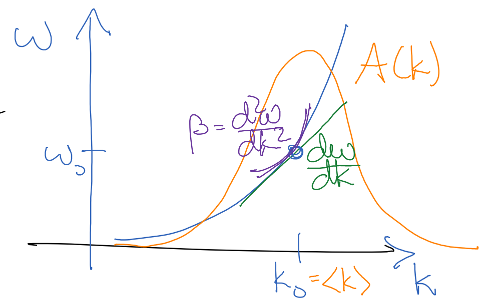
L10-Propagation of a Wave Packet: Dispersion

Friday, September 16, 2016 08:00

* How does a wave packet evolve? Add the time dependence:

$e^{ikx} \to e^{i(kx-\omega t)}$ of each frequency component, and

Taylor expand the dispersion relation



$$\omega(k) \approx \underbrace{\omega_0}_{\omega(k_0)} + \frac{d\omega}{dk}\Big|_{k_0}(k-k_0) + \frac{1}{2}\frac{d^2\omega}{dk^2}\Big|_{k_0}(k-k_0)^2$$

$$kx - \omega t = k_0 x + (k-k_0)x - \omega(k)t$$

$$\approx k_0\Big(x - \underbrace{\frac{\omega_0}{k_0}}_{v_\phi}t\Big) + (k-k_0)\Big(x - \underbrace{\frac{d\omega}{dk}\Big|_{k_0}}_{v_g}t\Big) - \frac{1}{2}(k-k_0)^2\underbrace{\frac{d^2\omega}{dk^2}\Big|_{k_0}}_{\beta}t$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\, A(k)\, e^{i(kx-\omega(k)t)}$$ (adding the time dependence)

$$\approx e^{i\underbrace{k_0(x-v_\phi t)}_{\phi}}\cdot\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dq\, A(k_0+q)\, e^{i\underbrace{q(x-v_g t)}_{\text{translation}} - \underbrace{\frac{i}{2}q^2\beta t}_{\text{broadening}}}}_{\text{envelope (wave packet)}}$$ where $q = (k-k_0)$

phase

The ripples travel at the phase velocity.
The "particle" is represented by the envelope, a function of x_t, which thus travels at the group velocity.
The dispersion dephases the Fourier components on the fringe of the wave packet, which causes it to spread out.

* Example: Gaussian wave packet: recall the Fourier Transform pair

$$\psi(x) = \underbrace{\frac{1}{\sqrt{\sqrt{2\pi}\Delta x}}}_{N_x = \text{norm}}\underbrace{e^{-\frac{1}{4}\left(\frac{x-x_0}{\Delta x}\right)^2}}_{\text{envelope}}\underbrace{e^{ik_0x}}_{\text{phase}} \Longleftrightarrow \underbrace{\frac{1}{\sqrt{\sqrt{2\pi}\Delta k}}}_{N_k = \text{norm}}\underbrace{e^{-\boxed{\frac{1}{4}}\left(\frac{k-k_0}{\Delta k}\right)^2}}_{\text{envelope}}\underbrace{e^{-i\boxed{x_0}k}}_{\text{phase}} = A(k)$$

DIFFERENCE!

$$= \frac{1}{\sqrt{2\pi}}\int dk\, A(k)\, e^{ikx} \qquad x \leftrightarrow k \qquad \frac{1}{\sqrt{2\pi}}\int dx\, \psi(x)\, e^{-ikx} =$$

DIFFERENCE!

Center the momentum: $A(k_0+q) = N_k e^{\underbrace{-\frac{1}{4}\frac{q^2}{\Delta k^2}}_{\text{gaussian}} - \underbrace{iqx_0}_{\text{shift}} - \underbrace{ik_0x_0}_{\text{global phase}}}$

$$\Psi(x,t) = e^{i(\phi - k_0x_0)}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dq\, N_k e^{-q^2\Big(\underbrace{\boxed{\frac{1}{4\Delta k^2}}}_{\Delta x^2(0)} + \underbrace{\boxed{\frac{i}{2}\beta t}}_{\Delta x^2(t)}\Big)} e^{iq(x - \underbrace{\boxed{(x_0+v_g t)}}_{x_0(t)})}$$ apply the above F.T. pair:

$$= e^{i\underbrace{k_0(x-v_\phi t)}_{\phi}}\, e^{-ik_0x_0}\, \frac{1}{\sqrt{\sqrt{2\pi}\Delta x(t)}}\, e^{-\frac{1}{4}\left(\frac{\boxed{x-x_0(t)}}{\boxed{\Delta x^2(t)}}\right)^2}$$ where $x_0(t) = x_0 + v_g t$, $\Delta x^2(t) = \Delta x^2(0) + \frac{i}{2}\beta t$

$$= e^{i\underbrace{k_0(x - v_\phi t)}_{\phi}}\, e^{-ik_0 x_0} \frac{1}{\sqrt{2\pi}\,\Delta x(t)}\, e^{-\frac{1}{4}\underbrace{\boxed{\frac{(x - x_0(t))^2}{\Delta x^2(t)}}}_{g}} \quad \text{where} \quad \begin{aligned} x_0(t) &= x_0 + v_g t \\ \Delta x^2(t) &= \Delta x^2(0) + \tfrac{i}{2}\beta t \end{aligned}$$

To understand the dispersion (spreading out of the wave packet), calculate the probability density $P(x) = |\psi(x)|^2$

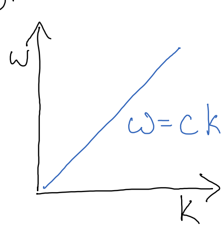
$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\,\Delta x(t)} \left| e^{-\frac{1}{4}\frac{(x-x_0)^2}{\Delta x^2(t)}} \right|^2 \qquad |e^{1/z}|^2 = e^{1/z} \cdot e^{1/z^*} = e^{\frac{z^* + z}{|z|^2}} = e^{2\,\mathrm{Re}[z]/|z|^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\Delta x(t)} e^{-\frac{1}{2}(x-x_0)^2 \mathrm{Re}[\Delta x^2(t)]/|\Delta x^2(t)|^2} = \frac{1}{\sqrt{2\pi}\,\Delta x(t)} e^{-\frac{1}{2}\frac{(x-x_0)^2}{\Delta x^2(0) + \beta^2 t^2/\Delta x^2(0)}}$$

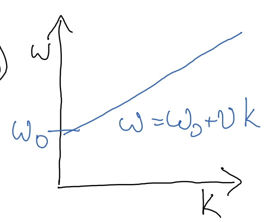
is a Gaussian distribution of width $\Delta x(t) = \Delta x(0)\sqrt{1 + \frac{1}{4}\beta^2 t^2/\Delta x^4(0)}$, which spreads out in time! (dispersion)

Thus the dispersion constant $\beta = \frac{d^2\omega}{dk^2}$, ie the second derivative of the dispersion relation, is actually what causes dispersion. The bending of light at different angles is due to frequency-dependent impedance $Z(k) = \sqrt{\mu/\varepsilon} = \mu v = \mu\omega/k$, which is related to dispersion.

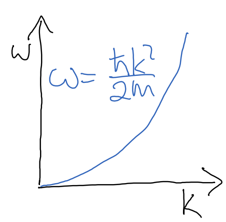
* Types of dispersion relations:

a) 

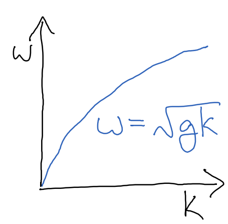
nondispersive medium [E&M vacuum] constant $v_\phi = v_g = c$

b) 

constant v_g with extra global phase factor $e^{i\omega_0 t}$

c) 

free particle: $2v_\phi = v_g = \frac{\hbar k}{m} = v_p$ ripples lag behind.

d) 

gravity waves: $\frac{1}{2}v_\phi = v_g = \frac{1}{2}\sqrt{g/k}$ ripples advance