L17-SHO Frobenius Method

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* Overview: we are going to solve the same problem using the Frobenius method (series solutions). Here are the main steps:

1) dimensionless $\mathcal{H}$: $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right)\psi=E\psi \Rightarrow \left(-\frac{d^2}{d\xi^2}+\xi^2-K\right)\psi=0$

2) asymptotic form: $\psi=h(\xi)e^{-\xi^2/2}$ $\quad h''-2\xi h'+(K-1)h=0$

3) power series: $h(\xi)=\sum_{j=0}^{\infty}a_j\xi^j$ $\quad a_{j+2}=\frac{2j+1-K}{(j+1)(j+2)}$

4) B.C.'s (truncation): quantization $E_n=\frac{1}{2}\hbar\omega K_n=\hbar\omega(n+\frac{1}{2})$ $\quad a_{j+2}^{(n)}=\frac{-2(n-j)}{(j+1)(j+2)}$

$$\psi_n(\xi)=\left(\frac{m\omega}{\pi\hbar}\right)\frac{1}{\sqrt{2^n n!}}H_n(\xi)\,e^{-\xi^2/2}$$

* TISE for SHO : $\hat{H}\psi=E\psi$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right)\psi(x)=E\,\psi(x)$$

a) solve ODE
b) apply B.C.'s.

- dimensionless variables:

let $\xi=\sqrt{\frac{m\omega}{\hbar}}x$ $\quad K=\frac{2E}{\hbar\omega}$ $\Rightarrow$ $\frac{d^2\psi}{d\xi^2}-(\xi^2-K)\psi=0$

- asymptotic limit: if $\xi^2\gg K$ then $\frac{d^2\psi}{d\xi^2}=\xi^2\psi$, $\psi\approx e^{-\xi^2/2}$

$$\frac{d}{d\xi}\left(\frac{d}{d\xi}e^{-\xi^2/2}\right)=\frac{d}{d\xi}\left(-\xi e^{-\xi^2/2}\right)=\underbrace{\cancel{-1}\cdot e^{-\xi^2/2}}_{\text{smaller}}+\underbrace{(-\xi)(-\xi)e^{-\xi^2/2}}_{\text{dominant as }\xi\to\infty}\approx\xi^2e^{-\xi^2/2}$$

let $\psi(\xi)=h(\xi)\,e^{-\xi^2/2}$ to factor out this dependence

$$\psi'=h'e^{-\xi^2/2}+h(-\xi)e^{-\xi^2/2}$$
$$\psi''=h''e^{-\xi^2/2}+2h'(-\xi)e^{-\xi^2/2}-he^{-\xi^2/2}+h\xi^2e^{-\xi^2/2}$$
$$\psi''-(\xi^2-K)\psi=\left(h''-2h'\xi+h\xi^2+(K-1)h\right)e^{-\xi^2/2}=0$$

thus $h''-2\xi h'+(K-1)h=0$ $\quad$ Hermite ODE with $K+1\to 2n$

- Power series solution: let $h = \sum_{j=0}^{\infty} a_j \xi^j$ $\quad h' = \sum_{j=0}^{\infty} a_j\, j\, \xi^{j-1}$

plug these into ODE: $\quad h'' = \sum_{j=0}^{\infty} a_j\, j(j-1)\, \xi^{j-2} = \sum_{j=0}^{\infty} (j+1)(j+2)\, a_{j+2}\, \xi^j$

$$\sum_{j=0}^{\infty} \left[(j+1)(j+2)\, a_{j+2} - 2j\, a_j + (K-1)\, a_j \right] \xi^j = 0 \qquad a_{j+2} = \frac{2j+1-K}{(j+1)(j+2)}\, a_j$$

solution: $h(\xi) = \Big[h_{even} = \underbrace{a_0}_{\text{normalization}} + \underbrace{a_2 \xi^2 + a_4 \xi^4 + \dots}_{\text{by recursion}} \Big] + \Big[h_{odd} = \underbrace{a_1}_{\text{norm}} \xi + \underbrace{a_3 \xi^3 + \dots}_{\text{recursion}} \Big]$

- quantization: $a_{j+2} \approx \frac{1}{j/2}\, a_j \approx \frac{a_0}{(j/2)!}$ $\quad$ exponential growth!

$$h(\xi) \approx a_0 \sum_{j=0}^{\infty} \frac{1}{(j/2)!} \xi^j \approx a_0 \sum_j \frac{1}{j!}\, \xi^{2j} \approx a_0\, e^{+\xi^2} \quad \text{blows up.}$$

only get normalized solution if series terminates

if $K_n = 2n+1$ then $a_{n+2} = a_{n+4} = \dots = 0$. $\quad K_n$ satisfies B.C.

thus $E_n = \frac{\hbar\omega}{2}(2n+1) = \hbar\omega(n+\tfrac{1}{2})$ $\qquad a_{j+2}^{(n)} = \frac{-2(n-j)}{(j+1)(j+2)}$

- Hermite polynomials: "normalized" so $a_n^{(n)} = 2^n$

$h_0(\xi) = a_0^{(0)} = 1$ $\quad \boxed{H_0(\xi) = 1}$ $\qquad h_1(\xi) = a_1^{(1)} = \xi$ $\quad \boxed{H_1(\xi) = 2\xi}$

$h_2(\xi) = a_0^{(2)} + a_2^{(2)} \xi^2 \propto 1 - 2\xi$ $\qquad h_3(\xi) = a_1^{(3)} \xi + a_3^{(3)} \xi^3 \propto \xi - \tfrac{2}{3}\xi^3$

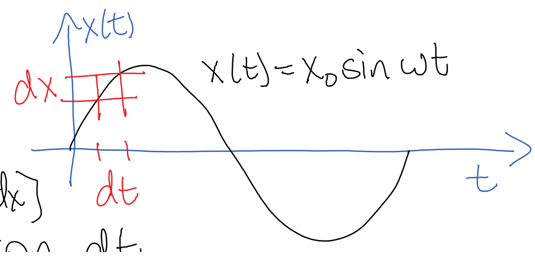
$\boxed{H_2(\xi) = 4\xi^2 - 2}$ $\qquad \boxed{H_3(\xi) = 8\xi^3 - 12\xi}$

- Normalized wave functions:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}}\, H_n(\xi)\, e^{-\xi^2/2}$$

$$\int_{-\infty}^{\infty} \psi_n(x)\, \psi_m(x)\, dx = \frac{1}{\sqrt{\pi\, 2^n n!\, 2^m m!}} \int_{-\infty}^{\infty} \underbrace{e^{-\xi^2}}_{\text{weight } w(\xi)} d\xi\; H_n(\xi)\, H_m(\xi) = \delta_{nm}$$

- Classical probability density:



$\rho\, dx$ = probability that $x \in [x, x+dx]$

$\propto$ time spent in this region dt.

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thus $\rho(x) \propto \frac{1}{dx/dt} = \frac{1}{\cos(\omega t)} = \frac{1}{\sqrt{1-\sin^2\omega t}} = \frac{1}{\sqrt{1-(x/x_0)^2}}$

$\int_{-x_0}^{x_0} \rho(x)\, dx = \int_{-x_0}^{x_0} \frac{k\,dx}{\sqrt{1-(x/x_0)^2}} = k x_0 \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = k x_0 \pi/2 = 1 \qquad u = x/x_0$

thus $\rho(x) = \frac{2/\pi}{\sqrt{x_0^2 - x^2}}$

Compare with $|\psi_{100}(x)|^2$ to see classical limit.

(Griffiths)

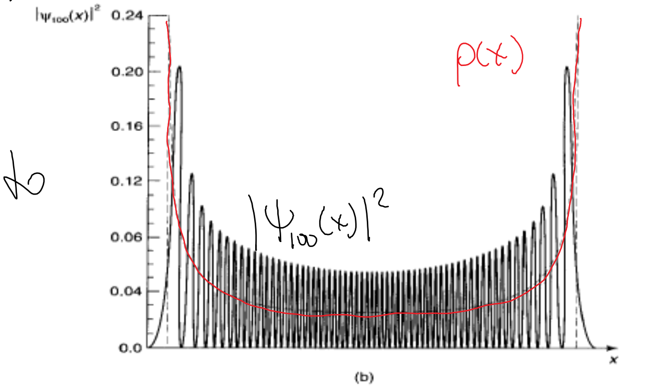

Figure 2.5: (a) The first four stationary states of the harmonic oscillator. (b) Graph of $|\psi_{100}|^2$, with the classical distribution (dashed curve) superimposed.