

L23-Vectors, Duals, Metric, Adjoint

Saturday, September 19, 2015

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* Vectors

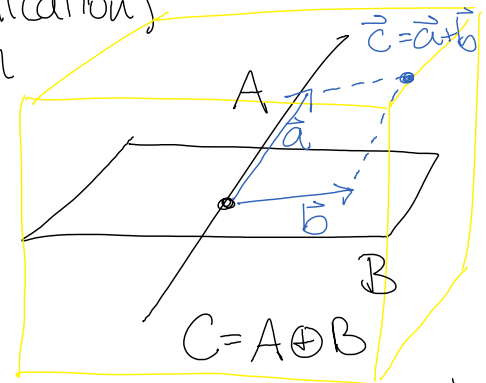
- prototype: arrows add head-to-tail, stretch column vectors of components
- formal: objects with **linear combinations**
 $\alpha \vec{u} + \beta \vec{v} + \dots = \sum \alpha_i \vec{v}_i \rightarrow \alpha_i \vec{v}_i$ (index notation: implied summation)
 - properties of scalar product, addition ensure this

- strange examples:

- apples & bananas & cherries $i \vec{a} + j \vec{b} + k \vec{c}$
- functions: $(\alpha_i f_i)(x) = \alpha_i f_i(x)$ notation: $\langle x | [\alpha_i | f_i] \rangle$
- complex numbers: $z = x + iy$ basis: $1, i$
- matrices: (without matrix multiplication)

- **subspaces**: **closed** under addition

- subplanes, sublines, etc,
all through the origin
- **direct sum** of subspaces $C = A \oplus B$
 $\vec{c} = \vec{a} + \vec{b} \sim (\vec{a}, \vec{b})$



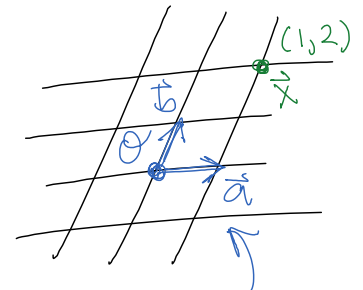
- $\vec{a}, \vec{b}$ are the **projection** of $\vec{c}$ into these subspaces (not orthogonal! no metric)
- ## - **basis**: break space down to line subspaces (1-d) and pick one vector on each line.

- projection to **components** (α, β, γ)

$$\vec{x} = \vec{a} \alpha + \vec{b} \beta + \vec{c} \gamma$$

$$= (\vec{a} \ \vec{b} \ \vec{c}) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

basis $\uparrow$ components $\leftarrow$



- structure of any vector space is characterized by its basis + egg crate extension
- a basis must be **linearly independent** and **complete**

- what does this buy us for Quantum Mechanics?
superposition / interference of complex amplitudes

- * inner product - metric, contraction
- adding structure to vector space as "multilinear extensions"

$$\vec{a} \cdot \vec{b} = a_x^* b_x + a_y^* b_y + \dots = (a_x^* a_y^* \dots) \begin{pmatrix} b_x \\ b_y \\ \vdots \end{pmatrix} = \vec{a}^T \vec{b}$$

• $\equiv$ T^* $\equiv$ $^+$

- properties:

- a) multilinear $(\alpha_i \vec{a}_i) \cdot \vec{b} = \alpha_i (\vec{a}_i \cdot \vec{b})$, also for $\beta_i \vec{b}_i$
- b) symmetric $\vec{a} \cdot \vec{b} = (\vec{b} \cdot \vec{a})^*$ (almost!)
- c) scalar valued - contracts two vectors to a scalar.

- vs outer product: $\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} (b_x b_y b_z) = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix}$ Trace.

- the trace of the outer product = inner product

- inner product of functions (as vectors):

$$\vec{a} \cdot \vec{b} = \sum_i a_i^* b_i \rightarrow \langle f | g \rangle = \int dx f^*(x) g(x)$$

$\uparrow$ index i $\uparrow$ index x .

- we will continue this extended analogy next week

- generic metric: $\vec{x} \cdot \vec{y} = (\vec{b}_i x^i) \cdot (\vec{b}_j y^j)$

$$\vec{x} \cdot \vec{y} = (\vec{b} \otimes)^T (\vec{b} \otimes) = \vec{x}^T \underbrace{\vec{b} \otimes \vec{b}}_G \vec{y} = (x^1 x^2 x^3) \begin{pmatrix} \vec{b}_1 \cdot \vec{b}_1 & \vec{b}_1 \cdot \vec{b}_2 & \vec{b}_1 \cdot \vec{b}_3 \\ \vec{b}_2 \cdot \vec{b}_1 & \vec{b}_2 \cdot \vec{b}_2 & \vec{b}_2 \cdot \vec{b}_3 \\ \vec{b}_3 \cdot \vec{b}_1 & \vec{b}_3 \cdot \vec{b}_2 & \vec{b}_3 \cdot \vec{b}_3 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}$$

$= \vec{x}^T G \vec{y} = g_{ij} x^i y^j$ $G \leftarrow$ metric $\rightarrow$

- a symmetric matrix in general.

- orthonormal basis: $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

(implies independence)

- useful for picking out components

$$\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix} \cdot (\hat{e}_1 \hat{e}_2 \hat{e}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}$$

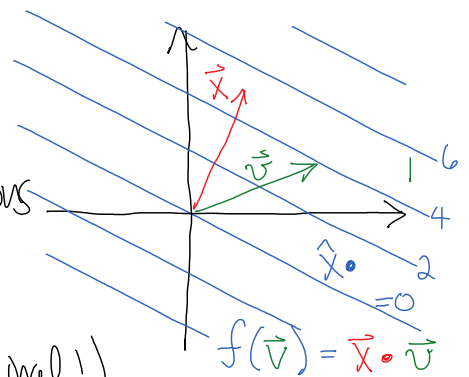
$$\hat{e} \cdot \vec{v} = \hat{e} \cdot (\hat{e} \cdot \vec{v}) = (\hat{e} \cdot \hat{e}) \cdot \vec{v} = \mathbb{I} \cdot \vec{v} = \vec{v}$$

- closure relationship $\vec{V} = \hat{E} \psi = \hat{E} \hat{E} \cdot \vec{V} = 1 \vec{V}$
 $\hat{E} \hat{E} \cdot = 1$ or $\sum_i \hat{E}_i \hat{E}_i \cdot = 1$ (implies completeness). $\leftarrow$ $1 = \text{identity on } V$

- orthogonal projections $P_i = \hat{E}_i \hat{E}_i \cdot$ note the $\cdot$!
 $P_x = \hat{x} \hat{x} \cdot \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (100) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $P_x^2 = P_x$ $P_x + P_y + P_z = I$

* Linear functionals (Forms): Dual Space (half of the inner product).

- the object $\vec{x} \cdot \sim (x_1^* x_2^* \dots)$ is a linear function of $\vec{V}$
- it is a vector itself under operations
 $[\alpha(\vec{x}_1 \cdot) + \beta(\vec{x}_2 \cdot)](\vec{y}) \equiv \alpha \vec{x}_1 \cdot \vec{y} + \beta \vec{x}_2 \cdot \vec{y}$
- this is called the dual space.
 (the dual of the dual is $\sim$ the original!)



- the cobasis is defined to be "orthonormal"
 $\tilde{E}_i(\tilde{E}_j) \equiv \delta_{ij}$ (can be defined even without a metric)

- can be used to extract co-ordinates:
 $\tilde{E}_i(\vec{x}) = \tilde{E}_i(\alpha_j \tilde{E}_j) = \alpha_j \tilde{E}_i(\tilde{E}_j) = \alpha_j \delta_{ij} = \alpha_i$
 $\vec{x}(\tilde{E}_i) = (\beta_j \tilde{E}_j)(\tilde{E}_i) = \beta_j \tilde{E}_j(\tilde{E}_i) = \beta_j \delta_{ji} = \beta_i$

* Adjoint: the metric provides a mapping between a vector space and its adjoint.

$$\vec{V} \rightarrow \tilde{V} \equiv \vec{V} \cdot \equiv \vec{V}^+ \quad \tilde{V}(\vec{x}) = \vec{V} \cdot \vec{x}$$

- for orthonormal basis:

$$\tilde{e}_i = \hat{e}_i \cdot \quad \text{since} \quad \tilde{e}_i(\tilde{e}_j) = \delta_{ij} = \hat{e}_i \cdot \hat{e}_j$$

$$\text{— thus } \vec{v}^\dagger = (\hat{e}_i v_i)^\dagger = (\hat{e}_i v_i) \cdot = v_i^* \hat{e}_i \cdot = v_i^* \tilde{e}_i$$

The adjoint of a vector is the conjugate transpose.
row vectors are waiting to gobble up column vectors.

$$\text{— same for functions: } \langle f | g \rangle = \int dx f^*(x) g(x)$$

bra: $\langle f | \sim \int dx f^*(x)$ a linear functional of functions
ket: $|g\rangle \sim \underbrace{g(x)}_{x^{\text{th}} \text{ component.}}$ a plain old function

— note that $\begin{matrix} \tilde{f} \tilde{g} \\ \vec{f} \vec{g} \end{matrix}$ or $\langle f | g \rangle$ is an inner product
 $\begin{matrix} \tilde{f} \tilde{g} \\ \vec{f} \vec{g} \end{matrix}$ or $|f\rangle \langle g|$ is an outer product.