

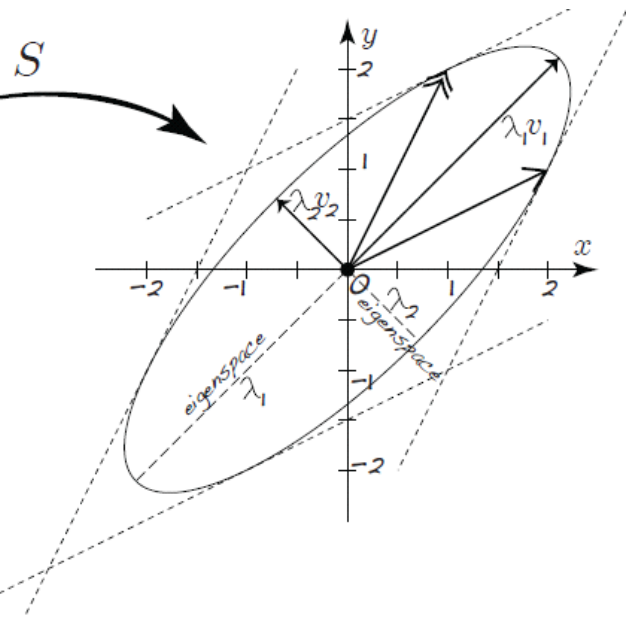
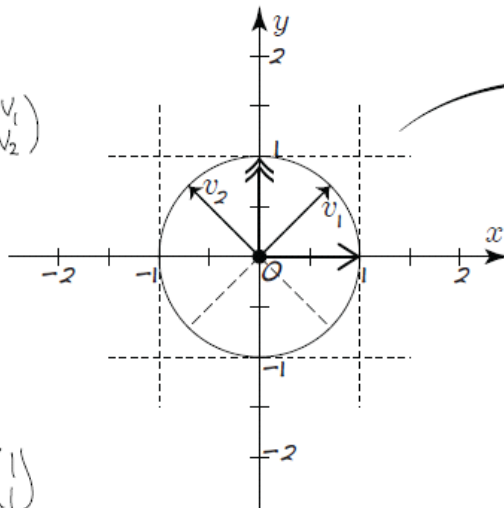
$$S_V = \lambda_V$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

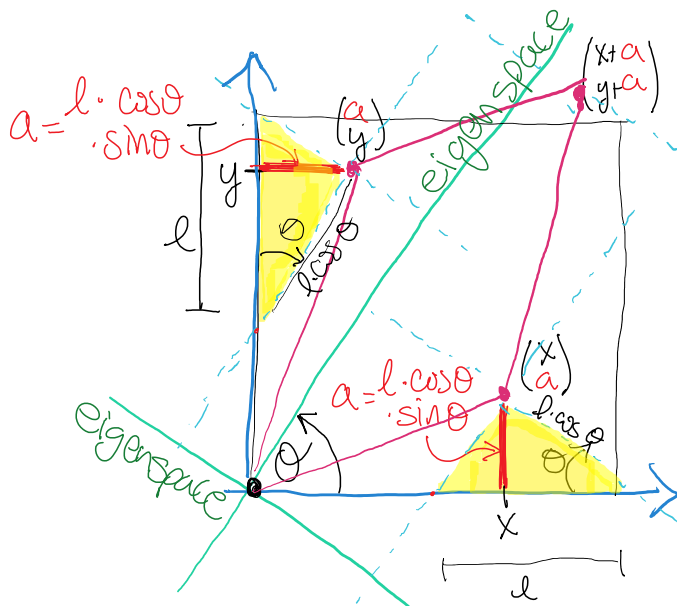
$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



- $$(x-x_0, y-y_0) \begin{pmatrix} \frac{1}{a^2} & \varepsilon \\ \varepsilon & \frac{1}{b^2} \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} = 1$$

- symmetric matrices are pure stretches



$$M = \begin{pmatrix} x & a \\ a & y \end{pmatrix}$$

- any shape of trapezoid can be made by flattening a square λ along θ line
- we still need to rotate to get an arbitrary trapezoid $M = RS$

* Eigensystem notation

- if $M^\dagger = M$ (Hermitian) then there exists a complete set of eigenvalues λ_i and eigenvectors $\vec{v}_i$ such that $M\vec{v}_i = \lambda_i\vec{v}_i$ and $\vec{v}_i^\dagger \vec{v}_j = \delta_{ij}$.

- we can put $\vec{v}_i$ together to form a unitary transformation matrix $V = (\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n)$,

so $MV = VD$ $D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}$ (diagonalization)

$$\begin{pmatrix} m_{xx} & m_{xy} \\ m_{yx} & m_{yy} \end{pmatrix} \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} = \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

- V is unitary: $V^\dagger V = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix} \cdot (\vec{v}_1, \vec{v}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$
(orthonormal)

- V is also closed: $VV^\dagger = (\vec{v}_1, \vec{v}_2) \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix} = \underbrace{\vec{v}_1 \vec{v}_1^\dagger}_{P_1} + \underbrace{\vec{v}_2 \vec{v}_2^\dagger}_{P_2} = 1$

- D = matrix elements of M in basis V : $D = V^\dagger M V = \langle V | M | V \rangle$
 $I = V^\dagger V$

- eigen value decomposition of M :

$$\begin{aligned} M &= V D V^\dagger = \sum_i \lambda_i P_i \\ 1 &= V V^\dagger = \sum_i P_i \end{aligned}$$

* Characterization of linear operators

a) ANY linear function has a singular value decomposition (SVD)

$$M = \overset{\text{symmetric}}{R} \overset{\text{eig.}}{S} = \overset{\text{symmetric}}{R} \overset{\text{eig.}}{V} W V^\dagger = U W V^\dagger \quad \text{where} \quad U^\dagger U = I_B$$

$$M_{A \rightarrow B} = \overset{\text{symmetric}}{R} \overset{\text{eig.}}{S} = \overset{\text{rotation}}{R} \overset{\text{2 rotations}}{V} \overset{\text{diag.}}{W} \overset{\text{combined rotation}}{V^T} = U W V^T \quad \text{where} \quad \begin{aligned} U^T U &= I_B \\ V^T V &= I_A \\ W &\text{ diagonal} \end{aligned}$$

eigenvalues λ_i of W : **singular values**
 V, U are eigen-basis of A, B respectively.

- b) ANY linear operator has n complex eigenvalues: λ_i
 (Fundamental Theorem of Algebra on characteristic equation)
 γ_i = geometric multiplicity = number of each eigenvalue
 μ_i = algebraic multiplicity = dimension of eigenspace
 - if $\mu_i = \gamma_i$ then the matrix is diagonalizable
 - otherwise it still has a Jordan decomposition $M = V \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} V^{-1}$
 note that V is not necessarily orthogonal or normal! $\rightarrow V^{-1}$, not V^T
 - this is a similarity transform

- c) ANY normal matrix M (ie. $M^\dagger M = M M^\dagger$)
 has a unitary diagonalization
 (orthonormal eigenvectors). $V^\dagger V = I$

- d) [anti] Hermitian operators have
 [imaginary] real eigenvalues.

$$\lambda_i^* v_i^\dagger v_j = v_i^\dagger H^\dagger v_j = v_i^\dagger H v_j = v_i^\dagger v_j \lambda_j$$

$$\text{if } i=j: (\lambda_i - \lambda_i^*) v_i^\dagger v_i = 0 \rightarrow \lambda_i \text{ real}$$

$$\text{if } \lambda_i \neq \lambda_j: (\lambda_i - \lambda_j^*) v_i^\dagger v_j = 0 \rightarrow v_i^\dagger v_j = 0 \text{ (unitary)}$$

* what does $\dagger$ have to do with operators?

Symmetric / antisymmetric vs. Symmetric / orthogonal decomposition

~ recall complex numbers $u = \rho + i\phi$ $\rho^* = \rho$ $(i\phi)^* = -i\phi$

$$e^u = e^{\rho + i\phi} = r e^{i\phi} \quad |e^{i\phi}|^2 = \bar{e}^{i\phi} e^{i\phi} = e^{i0} = 1$$

~ similar behaviour of symmetric / antisymmetric matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{pmatrix} + \begin{pmatrix} 0 & (b-c)/2 \\ (c-b)/2 & 0 \end{pmatrix} = T + A$$

$$K = e^M = 1 + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + \dots = e^{T+A} \neq e^T e^A \text{ in general.}$$

M arbitrary matrix
T symmetric
A antisymmetric
S symmetric
R orthogonal

$$S = e^T = e^{V W V^T} = V e^W V^{-1} \quad R = e^A \quad R^T R = (e^A)^T e^A = e^{A^T + A} = e^0 = I$$

$$\det(e^{\lambda_1} e^{\lambda_2} \dots) = e^{\lambda_1} e^{\lambda_2} \dots = e^{\lambda_1 + \lambda_2 + \dots} = e^{\text{tr}(\lambda_1 \lambda_2 \dots)} \quad \det e^A = e^{\text{tr} A} = e^0 = 1$$

if $e^{T+A} = e^T \cdot e^A$ [i.e. T, A commute], then
 $K = S \cdot R$ polar decomposition