L28-Hilbert Space

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* Hilbert space: infinite-dimensional vector space where all inner products converge

[where $\langle \phi | \psi \rangle = \sum_{n=0}^{\infty} a_n^* b_n$ or $\langle f | g \rangle = \int f^*(x) g(x)\, dx$ always finite]

Most theorems carry over from finite-dimensional spaces.

* Everything follows from the following principles & notation

	finite	Hilbert	functional
orthonormality	$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$	$\langle n \| m \rangle = \delta_{nm}$	$\langle x \| x' \rangle = \delta(x - x')$
closure	$\sum \hat{e}_i \hat{e}_i \cdot = I$	$\sum_n \|n\rangle\langle n\| = I$	$\int dx\, \|x\rangle\langle x\| = I$
components	$V_i = \hat{e}_i \cdot \vec{V}$	$c_n = \langle n \| \psi \rangle$	$f(x) = \langle x \| f \rangle$
matrix elements	$M_{ij} = \hat{e}_i \cdot M(e_j)$	$H_{mn} = \langle m \| H \| n \rangle$	$G(x, x') = \langle x \| G \| x' \rangle$

* Correspondence of Linear Algebra principles:

	$\mathbb{R}^n$ Finite Vectors	(discrete infinite) $\mathcal{H}$ Hilbert space.	(continuous infinite) $\mathcal{F}$ functional space
vector basis component	$\vec{V} = \hat{e}_i V_i$ $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$	$\|\psi\rangle = \sum_{n=0}^{\infty} \|\phi_n\rangle c_n$	$\|f\rangle = \int dx\, \|x\rangle f(x)$
inner product	$\vec{V} \cdot \vec{W} = V_i^* W_i$ $(x\,x\,x)\begin{pmatrix} x \\ x \\ x \end{pmatrix}$	$\langle \psi \| \chi \rangle = \sum_{n=0}^{\infty} c_n^* d_n$	$\langle f \| g \rangle = \int dx\, f^*(x) g(x)$
adjoint (dual) vector	$\vec{V} \cdot = V_i^* \hat{e}_i \cdot$ $(x\,x\,x)$	$\langle \psi \| = \sum_{n=0}^{\infty} c_n^* \langle \phi_n \|$	$\langle f \| = \int dx\, f^*(x) \langle x \|$
orthonormal (unitary) basis.	$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$	$\langle \phi_i \| \phi_j \rangle = \delta_{ij}$	$\langle k \| k' \rangle = \delta(k - k')$
completeness (closure) expansion	$\sum \hat{e}_i \hat{e}_i \cdot = I$	$\sum_n \|\phi_i\rangle\langle \phi_i\| = I$	$\int dk\, \|k\rangle\langle k\| = I$
operator components	$\vec{W} = A \vec{V}$ $W_i = A_{ij} V_j$	$\|\chi\rangle = A \|\psi\rangle$ $d_m = A_{mn} c_n$	$\|g\rangle = A \|f\rangle$ $g(x) = \int dx' A(x, x') f(x)$
Hermitian adjoint	$(A\vec{V})^\dagger = \vec{V}^\dagger A^\dagger$ $A^\dagger_{ij} = A^*_{ji}$	$\langle A \psi \| = \langle \psi \| A^\dagger = (A \|\psi\rangle)^\dagger$ $\langle \psi \| A \| \chi \rangle^* = \langle \chi \| A^\dagger \| \psi \rangle$	
identity transform	$V_i = R_{ij'} V_{j'}$	$\psi(x) = \sum_n \phi_n(x)\, c_n$	$\psi(x) = \int dk \frac{1}{\sqrt{2\pi}} e^{ikx} A(k)$

identity transform	$V_i = R_{ij'} V_{j'}$ $R_{ij'} = \hat{e}_i \cdot e_{j'}$	$\Psi(x) = \sum_n \phi_n(x)\, C_n$ $\phi_n(x) = \langle x \mid \phi_n \rangle$	$\Psi(x) = \int dk \frac{1}{\sqrt{2\pi}} e^{ikx} A(k)$ $\frac{1}{\sqrt{2\pi}} e^{ikx} = \langle x \mid k \rangle$
orthogonal transform	$R^\dagger_{i'k} R_{kj'} = \hat{e}_{i'} \cdot \hat{e}_j = \delta_{ij}$ $R^\dagger = R^{-1}$	$\sum_n \phi_n^*(x)\phi(x') = \delta(x-x')$ $\int dx\, \phi_m^*(x)\, \phi_n(x) = \delta_{mn}$	$\frac{1}{2\pi}\int dk\, e^{-ikx'} e^{ikx} = \delta(x-x')$
Hermitian transform	$A^\dagger = A \quad Av = \lambda v$ $V^\dagger V = I \quad V^\dagger A V = D$	$\mathcal{H}^\dagger = \mathcal{H} \quad \mathcal{H}\lvert\phi_n\rangle = E\lvert\phi_n\rangle$ $\langle \phi_n \mid \phi_m \rangle = \delta_{nm} \quad \mathcal{H} = \sum_n \lvert\phi_n\rangle E_n \langle\phi_n\rvert$	$p = -i\hbar\partial_x \quad p^\dagger = p \quad p\lvert k\rangle = \hbar k\lvert k\rangle$ $\langle k' \mid k \rangle = \delta(k-k') \quad p = \int dk \lvert k'\rangle \hbar k \langle k\rvert$