EX1-Solution

Wednesday, October 19, 2016 09:49

University of Kentucky, Physics 520
Exam 1, 2016-10-20

Instructions: This exam is closed book—you may not consult any person or reference material except your formula sheet. Show intermediate work for partial credit. [80 pts maximum]

[10 pts] 1. A specific mode of blackbody radiation of frequency ω has energy quantized into steps: $0, \hbar\omega, 2\hbar\omega, 3\hbar\omega, \ldots$. The unnormalized probability for each of these energies follows the Boltzmann distribution $P(E_n) \propto e^{-\beta E_n}$, where $\beta = 1/kT$. Calculate the average energy of this mode. How is this different from the classical calculation, and how did it solve the ultraviolet catastrophe in blackbody radiation?

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = \frac{-\frac{d}{d\beta} Z}{Z}$$

$$Z = \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n = \frac{1}{1-e^{-\beta\hbar\omega}}$$

$$\frac{dZ}{d\beta} = \frac{-1}{(1-e^{-\beta\hbar\omega})^2} \cdot (-)(-\hbar\omega) e^{-\beta\hbar\omega}$$

$$\langle E \rangle = -\frac{dZ}{d\beta} \Big/ Z = \frac{\hbar\omega e^{-\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2} \Big/ \frac{1}{(1-e^{-\beta\hbar\omega})}$$

$$= \frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} = \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}$$

versus $\langle E \rangle = kT$ classically (integrating over all E).

$\langle E \rangle \to 0$ as $\omega \to \infty$ so that UV radiation doesn't dominate the spectrum.

[5 pts] 2. Compare and the contrast rows and columns of the table

E	ω
p	k

. Show the relations between quantities in each direction for a nonrelativistic free particle. Describe the principles involved and their historical origins. Show how this table naturally leads to the Time Dependent Schrödinger Equation (TDSE).

E, p are particle-like properties, while ω, k wave-like

Quantization: $E = \hbar\omega$ (Planck blackbody radiation quantization).
$p = \hbar k$ (deBroglie matter waves)

E, ω are time-like quantities, while p, k are space-like.

They are related by m, v : the dispersion relation.

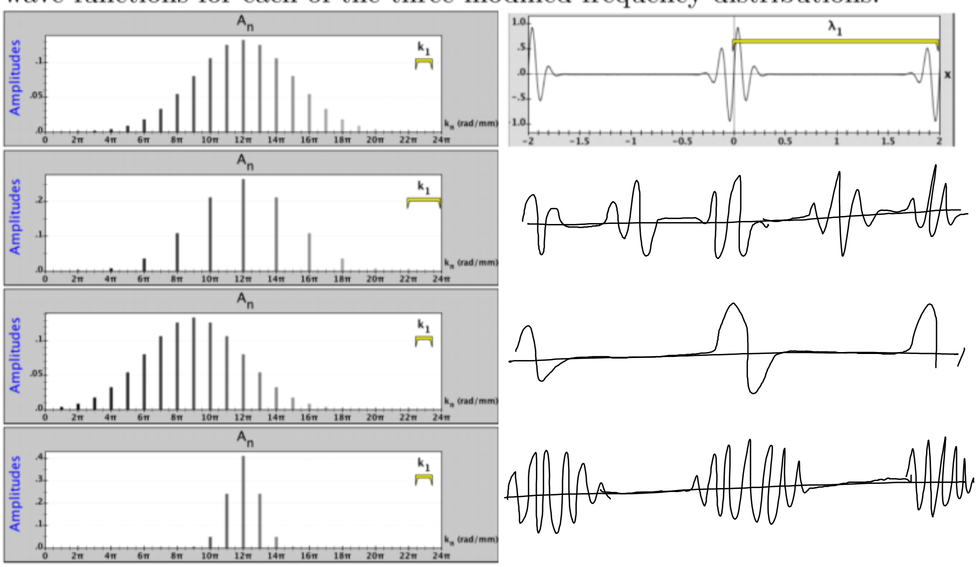
For a free particle, $E = \frac{p^2}{2m}$ $\quad \hbar\omega = \frac{\hbar^2 k^2}{2m}$

To get the wave equation, start with waves $\Psi = e^{i(kx-\omega t)}$

then $\partial_t \psi = -i\omega \psi$ and $\partial_x \psi = ik\psi$. Substitute the operators for the eigenvalues in the dispersion relation:

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t).$$

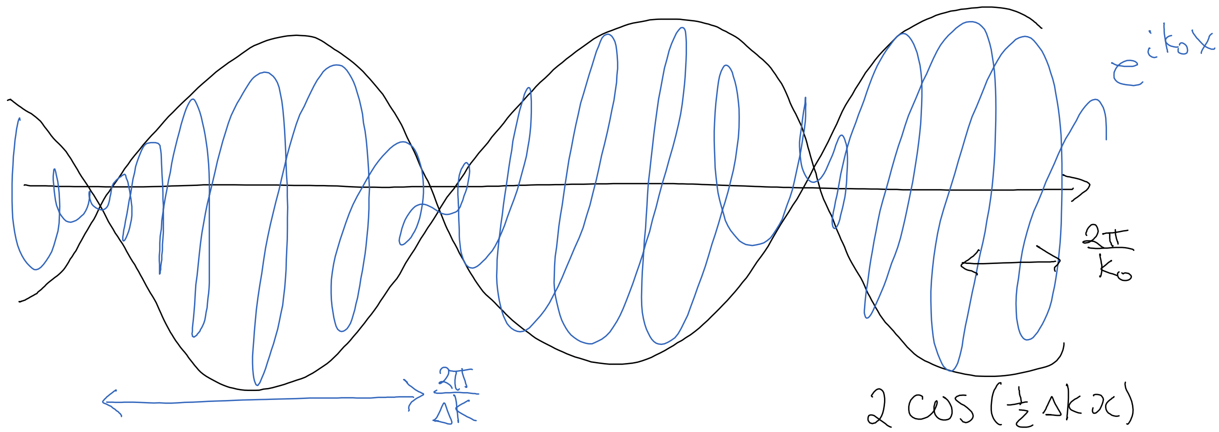
[5 pts] 3. Given the following wave function $\psi(x)$ with frequency component amplitudes A_k, draw wave functions for each of the three modified frequency distributions.



[5 pts] 4. a) Show that the function $\psi(x) = e^{ik_1x} + e^{ik_2x}$ is a pure frequency wave modulated by a sinusoidal envelope, and calculate the frequency of each feature.

$$\psi(x) = e^{ik_1x} + e^{ik_2x} = e^{i(k_0 - \frac{1}{2}\Delta k)x} + e^{(k_0 + \frac{1}{2}\Delta k)x} = e^{ik_0x}\left(e^{-\frac{1}{2}\Delta k x} + e^{+\frac{1}{2}\Delta k x}\right)$$

$$= e^{ik_0x}\cdot 2\cos\left(\tfrac{1}{2}\Delta k x\right) \qquad k_0 = \tfrac{1}{2}(k_1+k_2) \qquad \Delta k = k_2 - k_1$$



The envelope has frequency $\frac{\Delta k}{2}$ (bandwidth) and the phase k_0 (carrier).

[5 pts] b) Use the dispersion relation of the free particle to the add time dependence to both components. Calculate the velocity of both the pure frequency wave and of its sinusoidal envelope.

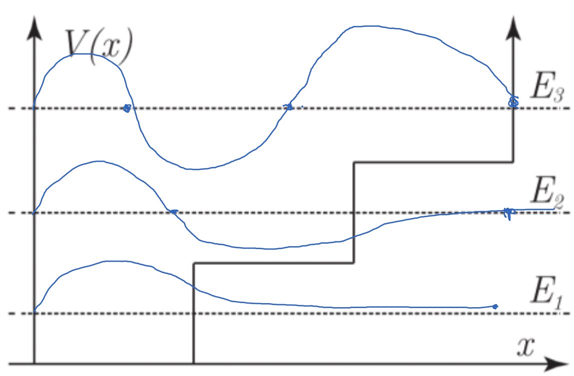
$$\Psi(x,t) = e^{i(k_1x-\omega_1t)} + e^{i(k_2x-\omega_2t)} \quad \text{where } \omega_1 = \frac{\hbar}{2m}k_1^2, \; \omega_2 = \frac{\hbar}{2m}k_2^2$$

$$= \underbrace{e^{i(k_0x-\omega_0t)}}_{\text{phase}} \cdot 2\cos\Big(\underbrace{\tfrac{1}{2}(\Delta k x - \Delta\omega t)}_{\text{group.}}\Big) \quad \text{as above.}$$

The phase velocity is $v_\phi = \frac{\omega_0}{k_0} = \frac{\hbar}{2m}\left(\frac{k_1^2+k_2^2}{k_1+k_2}\right)$

group velocity is $v_g = \frac{\Delta\omega}{\Delta k} = \frac{\hbar}{2m}\left(\frac{k_2^2-k_1^2}{k_2-k_1}\right) = \frac{\hbar}{2m}(k_1+k_2)$

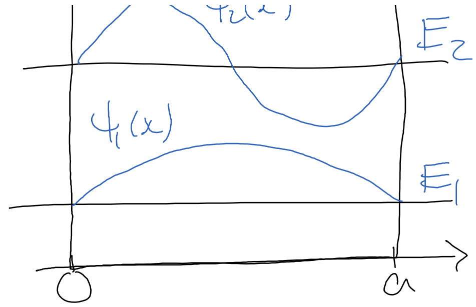
[5 pts] 5. Sketch the three lowest energy eigenfunctions of the "staircase potential":



[10 pts] 6. a) Solve for the normalized stationary states $\psi_n(x)$ and energies E_n of a particle in an infinite square well with extent $0 < x < a$.

$$\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \qquad \text{let } E = \frac{\hbar^2k^2}{2m}$$

$$\left(\frac{d^2}{dx^2} + k^2\right)\psi = 0$$



$$\left(\frac{d^2}{dx^2}+k^2\right)\psi=0$$

$$\psi = A\cos(kx) + B\sin(kx)$$

$$\psi(0)=0=A \Rightarrow A=0$$

$$\psi(a) = B\sin(ka)=0 \qquad k_n a = n\pi \qquad n=0,1,2\ldots$$

$$\psi_n(x) = B\sin(k_n x)$$

$$\langle\psi_n|\psi_n\rangle = B^2\int_0^a \sin^2(k_n x)\,dx = B^2\frac{a}{2}=1 \qquad B=\sqrt{\frac{2}{a}}$$

$$\text{thus } \psi_n(x)=\sqrt{\frac{2}{a}}\sin^2(k_n x) \qquad k_n=\frac{n\pi}{a} \qquad E_n=\frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2=\frac{h^2n^2}{8ma^2}$$

[5 pts] b) Given the initial wave function $\Psi(x,0)=\sqrt{2/a}$ if $x<a/2$ and 0 if $x>a/2$, calculate the first two amplitudes, and determine the time evolution $\Psi(x,t)$ of this state.

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar}$$

$$\Psi(x,0)=\sum_{n=1}^{\infty}c_n\psi_n(x) = \begin{cases}\sqrt{\frac{2}{a}} & \text{if } x<a/2\\ 0 & x>a/2\end{cases}$$

$$\langle\psi_{n'}|\Psi(x,0)\rangle = \sum_{n=1}^{\infty}c_n\langle\psi_{n'}|\psi_n\rangle = \sum_{n=1}^{\infty}c_n\delta_{n'n}=c_{n'}$$

$$=\int_0^{a/2}\sqrt{\frac{2}{a}}\sin(k_nx)\cdot\sqrt{\frac{2}{a}}\,dx \qquad \text{let } \theta=k_nx$$

$$d\theta = k_n\,dx$$

$$=\frac{2}{ak_n}\int_0^{n\pi/2}\sin\theta\,d\theta \qquad k_n\frac{a}{2}=\frac{n\pi}{a}\frac{a}{2}=\frac{n\pi}{2}$$

$$c_1=\frac{2}{\pi} \qquad c_2=\frac{2}{\pi}$$

$$\Psi(x,t)=\frac{2}{\pi}\psi_1(x)e^{-iE_1t/\hbar}+\frac{2}{\pi}\psi_2(x)e^{-iE_2t/\hbar}$$

[5 pts] 7. a) Explain what operators are and how they are used in quantum mechanics.

Operators are "functions of functions." They act on a

Operators are "functions of functions." They act on a function and change it into a new function.
In QM, operators represent quantities we can measure. They act on the wave function, to pull out information about that obersable, given the current state.
If the wave function is an eigenfunction of the operator, then then then the eigenvalue is the value of the observable.

[5 pts] b) Show that the commutator of x and $p = -i\hbar\frac{\partial}{\partial x}$ is $[x, p] = i\hbar$.

$$[x,p]\psi = x\left(-i\hbar\frac{\partial}{\partial x}\right)\psi - \left(-i\hbar\frac{\partial}{\partial x}\right)x\psi$$

$$= -i\hbar x\psi' + i\hbar(\psi + x\psi') = i\hbar\psi$$

thus $[x,p] = i\hbar$

[5 pts] b) Calculate $[a_+, a_-]$, where $a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$.

$$[a_+, a_-] = \left[\tfrac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x), \tfrac{1}{\sqrt{2\hbar m\omega}}(+ip + m\omega x)\right]$$

$$= \frac{1}{2\hbar m\omega}\left(\cancel{[-ip, +ip]}^{0} + [-ip, m\omega x] + [m\omega x, ip] + \cancel{[m\omega x, m\omega x]}^{0}\right)$$

$$= \frac{1}{2\hbar m\omega}\left(-im\omega(-i\hbar) + im\omega(i\hbar)\right) = -1$$

[10 pts] 8. Solve the equation $y' = \alpha y$ with initial condition $y(0) = y_0$ using the series method.

$$y = \sum_{j=0}^{\infty} a_j x^j \qquad y' = \sum_{j=0}^{\infty} a_j (j) x^{j-1} = \sum_{j=0}^{\infty} a_{j+1}(j+1)x^j$$

$$y' - \alpha y = \sum_{j=0}^{\infty}\left[a_{j+1}(j+1) - \alpha a_j\right]x^j = 0$$

$$a_{j+1} = \frac{\alpha}{j+1}a_j \qquad a_j = \frac{\alpha}{j}a_{j-1} = \frac{\alpha^2}{j(j-1)}a_{j-2} = \cdots = \frac{\alpha^j}{j!}a_0$$

$$y = a_0\sum_{j=0}^{\infty}\frac{(\alpha x)^j}{j!} = a_0 e^{\alpha x} \qquad y(0) = a_0 = y_0$$

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$$y = y_0 e^{\alpha x}$$