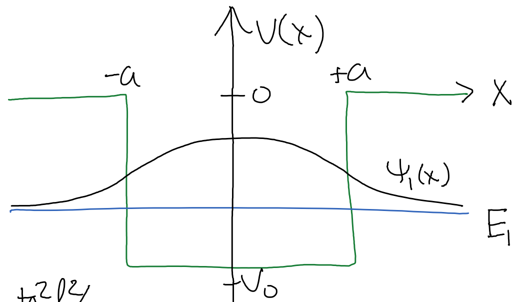
EX2-Solution

Friday, November 18, 2016 05:44

University of Kentucky, Physics 520
Exam 2, 2016-11-17

Instructions: This exam is closed book. A formula sheet with fundamental equations is allowed. Show intermediate work for partial credit. Do either 1 or 1′, but not both. [60 pts maximum]

[20 pts] 1. Calculate the energy of the ground state of a finite square well of width $2a$ and depth $V_0 = \hbar^2/2ma^2$, ie. $V(x) = -V_0$ if $|x| < a$ and 0 otherwise. Leave your answer as the graphical intersection of two curves, indicating the approximate value of energy in units of $\hbar^2/ma^2$.



$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V\psi = E\psi$$

6 $x<a$: $\psi_I = Ae^{\kappa x} + Be^{-\kappa x}$

$|x|<a$: $\psi_{II} = C\cos(kl) + D\sin(kl)$ $\quad E+V_0 = \hbar^2 l^2/2m$

$x>a$: $\psi_{III} = Fe^{-\kappa x} + Ge^{\kappa x}$ $\quad E = -\hbar^2\kappa^2/2m$

8 symmetry: $\psi(x) = \psi(-x)$ ground state $D=0$

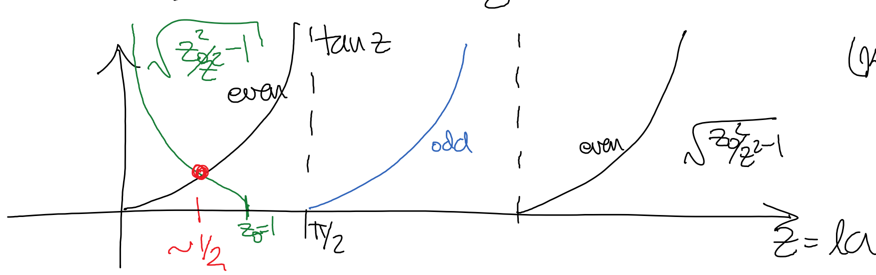
B.C.: $\psi_{III}(\infty) = 0 \rightarrow G = 0$

$\psi_{II}(a) = \psi_{III}(a)$: $C\cos(la) = Fe^{-\kappa a}$

$\psi'_{II}(a) = \psi'_{III}(a)$: $-l\cdot C\cdot\sin(la) = -\kappa Fe^{-\kappa a}$

6 $\tan(la) = \kappa/l = \sqrt{\frac{z_0^2 - z^2}{z^2}} = \sqrt{z_0^2/z^2 - 1}$ $\quad -\frac{\hbar^2\kappa^2}{2m} + V_0 = \frac{\hbar^2 l^2}{2m}$

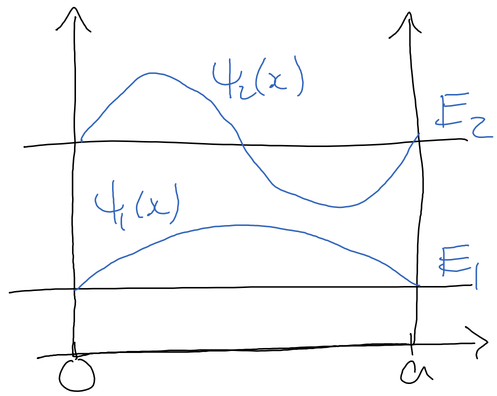
$$(\kappa a)^2 = \underbrace{\frac{2mV_0a^2}{\hbar^2}}_{z_0^2=1} - \underbrace{(la)^2}_{z^2}$$



$$E_1 = -V_0 + \frac{\hbar^2 l^2}{2m} = -V_0 + \frac{\hbar^2 z^2}{2ma^2} = \frac{\hbar^2}{2ma^2}\left(-1 + \frac{\sim 1}{4}\right) \approx \frac{-3\hbar^2}{8ma^2}$$

[5 pts] 1′ a). Solve for the normalized stationary states $\psi_n(x)$ and energies E_n of a particle in an infinite square well of extent $0 < x < a$.

$$\frac{\hbar^2}{2m}\frac{d^2}{dx}\psi = E\psi \qquad \text{let } E = \frac{\hbar^2 k^2}{2m}$$



$$\left(\frac{d^2}{dx^2} + k^2\right)\psi = 0$$

1 $\psi = A\cos(kx) + B\sin(kx)$

$\psi(0) = 0 = A \Rightarrow A = 0$

2 $\psi(a) = B\sin(ka) = 0 \qquad k_n a = n\pi \qquad n = 0, 1, 2 \ldots$

$\psi_n(x) = B\sin(k_n x)$

$\langle\psi_n|\psi_n\rangle = B^2 \int_0^a \sin^2(k_n x)\,dx = B^2 \frac{a}{2} = 1 \qquad B = \sqrt{\frac{2}{a}}$

2 thus $\psi_n(x) = \sqrt{\frac{2}{a}}\sin^2(k_n x) \qquad k_n = \frac{n\pi}{a} \qquad E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 = \frac{h^2 n^2}{8ma^2}$

[5 pts] b) Given the initial wave function $\Psi(x, 0) = \sqrt{2/a}$ if $x < a/2$ and 0 if $x > a/2$, calculate the first two amplitudes, and determine the time evolution $\Psi(x, t)$ of this state.

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} & \text{if } x < a/2 \\ 0 & x > a/2 \end{cases}$$

$$\langle\psi_{n'}|\Psi(x,0)\rangle = \sum_{n=1}^{\infty} c_n \langle\psi_{n'}|\psi_n\rangle = \sum_{n=1}^{\infty} c_n \delta_{n'n} = c_{n'}$$

$$= \int_0^{a/2} \sqrt{\frac{2}{a}}\sin(k_n x)\cdot\sqrt{\frac{2}{a}}\,dx \qquad \text{let } \theta = k_n x$$

$$d\theta = k_n dx$$

3 $$= \frac{2}{ak_n}\int_0^{n\pi/2} \sin\theta\, d\theta \qquad k_n \frac{a}{2} = \frac{n\pi}{a}\frac{a}{2} = \frac{n\pi}{2}$$

$$c_1 = \frac{2}{\pi} \qquad c_2 = \frac{2}{\pi}$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

2 $$= \frac{2}{\pi}\psi_1(x)e^{-iE_1t/\hbar} + \frac{2}{\pi}\psi_2(x)e^{-iE_2t/\hbar}$$

[5 pts] c) Calculate $\langle E \rangle$ for the wave function in part b) using the first two amplitudes.

$$\langle E \rangle = \begin{pmatrix} c_1^* & c_2^* & \dots \end{pmatrix} \begin{pmatrix} E_1 & & \\ & E_2 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 2/\pi & 2/\pi \end{pmatrix} \begin{pmatrix} h^2/8ma^2 & \\ & \frac{4h^2}{8ma^2} \end{pmatrix} \begin{pmatrix} 2/\pi \\ 2/\pi \\ \vdots \end{pmatrix}$$

5 $$= \frac{4}{\pi^2}\cdot\frac{h^2}{8ma^2} + \frac{4}{\pi^2}\cdot\frac{h^2 4}{8ma^2} = \frac{5h^2}{2\pi^2 ma^2}$$

Since we only used 2 states, we could divide by the partial normalization:

$$\langle E \rangle = \frac{5h^2}{2\pi^2 ma^2} \Big/ \left(\frac{4}{\pi^2} + \frac{4}{\pi^2}\right) = \frac{5}{2}\,\frac{h^2}{8ma^2}$$

[20 pts] 2. Calculate the probability that a quantum particle of mass m and energy $E > V_0$ incident from the left ($x < 0$) will bounce back from a step potential barrier of height V_0, i.e. $V(x) = 0$ if $x < 0$ and V_0 if $x > 0$.

$$\psi_I = Ae^{ikx} + Be^{-ikx} \qquad \frac{\hbar^2 k^2}{2m} = E$$

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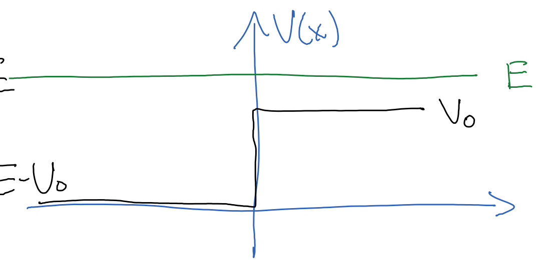
$$\psi_{II} = Fe^{ilx} + \overset{0}{\cancel{G e^{-ilx}}} \qquad \frac{\hbar^2 l^2}{2m} = E - V_0$$



$$\psi_I(0) = \psi_{II}(0): \quad A + B = F \qquad (1)$$

$$\psi_I'(0) = \psi_{II}'(0): \quad ikA - ikB = ilF \qquad (2)$$

8

$$ik\,(1) + (2): \quad 2ikA = i(k+l)F \qquad F = \frac{2k}{k+l}A$$

$$il\,(1) - (2): \quad i(l-k)A + i(l+k)B = 0 \qquad B = \frac{l-k}{l+k}A$$

prob. of bounce-back

4 $$R = \left|B/A\right|^2 = \left|\frac{l-k}{l+k}\right|^2 = \frac{\sqrt{E-V_0} - \sqrt{E}}{\sqrt{E-V_0} + \sqrt{E}}$$

3. A two-state system has the Hamiltonian $\hat{H} = \begin{pmatrix} 0 & -B \\ -B & 0 \end{pmatrix}$. The initial state is $|\Psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the position operator is $\hat{X} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Calculate:

[5 pts] a) The energy eigenvalues and eigenstates.

$\hat{H}|\psi\rangle = E|\psi\rangle \qquad \begin{pmatrix} 0 & -B \\ -B & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

3

$\begin{vmatrix} -E & -B \\ -B & -E \end{vmatrix} = E^2 - B^2 = 0 \qquad E_I = -B \quad E_{II} = +B$

eigenstate I: $\begin{pmatrix} B & -B \\ -B & B \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \psi_1 = \psi_2$

2 $\quad |I\rangle = n\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \langle I|I\rangle = |n|^2 \cdot 2 = 1 \qquad n = \frac{1}{\sqrt{2}}$

eigenstate II: $\begin{pmatrix} -B & -B \\ -B & -B \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \psi_1 = -\psi_2$

$|II\rangle = n\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \langle II|II\rangle = |n|^2 \cdot 2 = 1 \qquad n = \frac{1}{\sqrt{2}}$

thus $\quad |I\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \qquad |II\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$

[5 pts] b) the time evolution of the state: $|\Psi(t)\rangle$

$|\Psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_I|I\rangle + c_2|II\rangle$

$c_I = \langle I|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(1 \;\; 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$

2

$c_{II} = \langle II|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(1 \; -1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-1}{\sqrt{2}}$

$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}|I\rangle e^{iBt/\hbar} - \frac{1}{\sqrt{2}}|II\rangle e^{-iBt/\hbar}$

3 $\quad = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\right)e^{iBt/\hbar} - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)\right)e^{-iBt/\hbar}$

$= \frac{1}{2}\left(e^{iBt/\hbar} - e^{-iBt/\hbar}\right)|1\rangle + \frac{1}{2}\left(e^{iBt/\hbar} + e^{-iBt/\hbar}\right)|2\rangle$

$$= \underbrace{i\sin(Bt/\hbar)}_{c_1}\,|1\rangle + \underbrace{\cos(Bt/\hbar)}_{c_2}\,|2\rangle = \begin{pmatrix} i\sin Bt/\hbar \\ \cos Bt/\hbar \end{pmatrix}$$

[5 pts] c) the time-dependent expectation value $\langle \hat{X} \rangle(t)$

$$\langle X \rangle = P_1(t)\cdot x_1 + P_2(t)\cdot x_2$$

$$= \sin^2(Bt/\hbar)\cdot 2 + \cos^2(Bt/\hbar)\cdot 1$$

5

$$= 2 - \cos^2(Bt/\hbar) = 1 + \sin^2(Bt/\hbar)$$

$$= 3/2 - 1/2\cos(2Bt/\hbar)$$

[5 pts] d) the probability of measuring $X = 2$ and the probability of measuring $X = -2$ at time t.

3 $P(x=2)(t) = |c_1|^2 = \sin^2(Bt/\hbar)$

2 $P(x=-2)(t) = 0$ not an eigenstate of $\hat{X}$