EX1-Solution

Wednesday, October 4, 2017 04:02

[20 pts] 1. Calculate the classical expectation value $\langle E \rangle$ of energy of single mode of blackbody radiation, assuming the continuous Boltzmann distribution of energy $P(E) \propto e^{-\beta E}$, with the constant parameter $\beta = 1/kT$ for a given temperature T.

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$$\langle E \rangle = \frac{\int_0^\infty f(E)dE \cdot E}{\int_0^\infty f(E)dE}$$

$$\int_0^\infty f(E)dE \cdot E = \int_0^\infty e^{-\beta E} dE \cdot E = \frac{-1}{\beta}\int_0^\infty E \, d\, e^{-\beta E}$$

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$$= \frac{-1}{\beta}\left[E e^{-\beta E}\right]_0^\infty - \frac{-1}{\beta}\int_0^\infty e^{-\beta E} dE = \frac{1}{\beta}\int_0^\infty f(E)\,dE$$

thus $\langle E \rangle = \frac{1}{\beta} = kT$

Alternate: $Z \equiv \int_0^\infty f(E)dE = \int_0^\infty e^{-\beta E} dE = \frac{-1}{\beta} e^{-\beta E}\Big|_0^\infty = \frac{1}{\beta}$

$$\int_0^\infty f(E)dE \cdot E = \int_0^\infty e^{-\beta E} dE \cdot E = -\frac{\partial}{\partial \beta}\int_0^\infty e^{-\beta E} dE$$

$$= -\frac{\partial}{\partial \beta}\int_0^\infty f(E)dE = -\frac{\partial Z}{\partial \beta} = \frac{1}{\beta^2}$$

thus $E = -\partial_\beta \ln Z = \frac{-\partial Z/\partial\beta}{Z} = \frac{1/\beta^2}{1/\beta} = \frac{1}{\beta} = kT$

Explain why the number of modes radiation between frequency ν and $\nu + d\nu$ in a cubic cavity is proportional to $\nu^2 d\nu$. What is the ultraviolet catastrophe?

From boundary conditions, $n\lambda/2 = L$ and $\nu\lambda = c$

thus $\nu = \frac{c}{2L} n$ in each dimension.

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in 3-d, # of states $= 2\, d^3 n = 2\left(\frac{c}{2L}\right)^3 \cdot d^3\nu = \frac{\pi c}{V}\nu^2 d\nu$

ie: $\nu^2 d\nu$ is the volume of a shell in phase space.

The number of high frequency modes $\to \infty$ as $\nu \to \infty$

2 and each mode has $\langle E \rangle = kT$ of energy.

[20 pts] 2. Classify the different categories, principles, and relations contained vertically and horizontally in the table $\begin{array}{c|c} E & \omega \\ \hline p & k \end{array}$ for a nonrelativistic particle of mass m and velocity v, and describe the physical effects. Show how these principles lead naturally to the Time Dependent Schrödinger Equation (TDSE) as a wave equation.

quantization

particle $\Leftrightarrow$ wave

	particle	wave	
6	E	ω	time-like
			$\updownarrow$ dispersion
	p	k	space-like

$E = \hbar\omega \qquad E = \frac{p^2}{2m}$

$p = \hbar k \qquad \hbar\omega = \frac{\hbar^2 k^2}{2m}$

Quantization mixes the properties of waves and particles.

Dispersion describes the propagation of waves/particles:

2 $v_\phi = \frac{\omega}{k} = \frac{E}{p} = \frac{p}{2m} = \frac{1}{2} v$ (wave) $\qquad v_g = \frac{dE}{dm} = \frac{2p}{2m} = v$ (particle)

$\beta = \frac{d^2\omega}{dk^2} = \frac{\hbar}{m} \propto \frac{1}{m}$ (dispersion)

To get the wave equation let $\Psi = e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ $\quad \omega \to i\partial_t \quad k \to -i\nabla$

4 the dispersion relation becomes: $i\hbar\partial_t \Psi = \frac{-\hbar^2}{2m}\nabla^2 + V\Psi$,

adding kinetic + potential energy.

Compare and contrast classical waves versus particles. Explain how wave-particle duality leads to quantization of both particles [deBroglie] and waves [Planck]. Describe how matter behaves as both particles and waves in Quantum Mechanics.

particles: definite E, p, $x(t)$, interacts conserving E, p (localized), quantum of mass/charge.

4 waves: collective mode (quantized by boundary conditions) dispersion, interference, impedance, dispersion, polarization; energy propogates & spreads out.

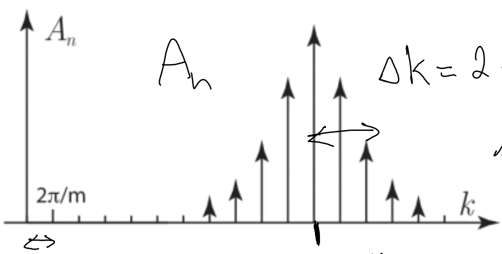
dispersion, interference, impedance, dispersion,
polarization; energy propogates & spreads out.

2 Planck quantized waves as "packets of energy"

de Broglie quantized particles as "waves with B.C.'s"

2 Everything propogates as waves (TDSE) and interacts (detected) as particles (Born interpretation).

[20 pts] 3. Given the following distribution A_n of component amplitudes of a free-particle wave, sketch the corresponding wave function $\psi(x)$. Identify the features associated with the a) fundamental frequency k_1, b) central frequency k_0, and c) bandwidth Δk in both A_n and $\psi(x)$.



A_n

$\Delta k = 2 \cdot \frac{2\pi}{L} = \frac{1}{2\Delta x}$

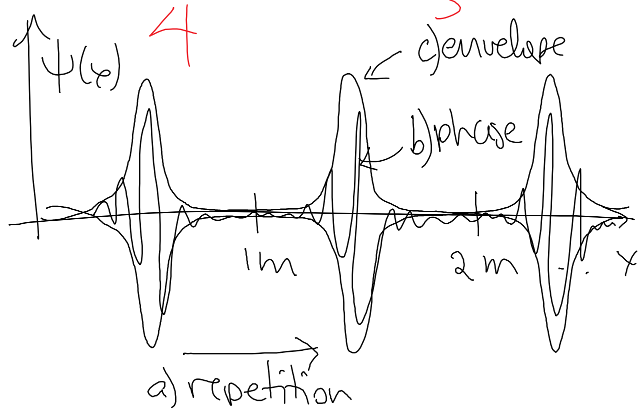
$\Delta x = \frac{L}{8\pi} = 0.04\,m$ (packet width)

$k_1 = \frac{2\pi}{L}$ $L = 1m$ (fundamental period)

$k_0 = 11 \cdot \frac{2\pi}{m} = \frac{2\pi}{\lambda}$ 3

$\lambda = \frac{1}{11}m = 0.091\,m$ (phase wavelength)

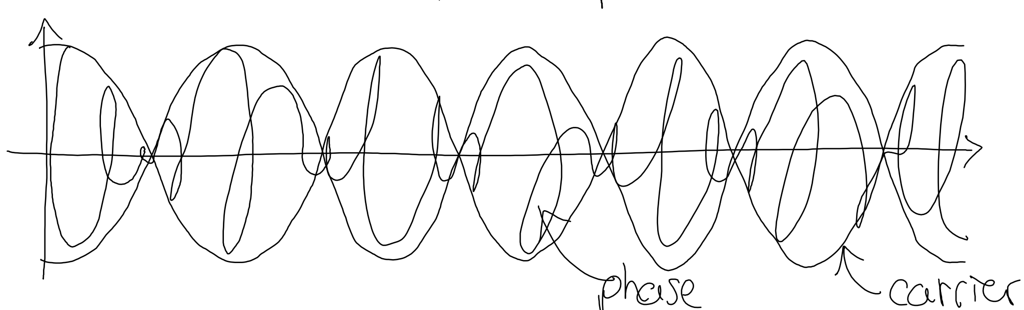
4 3



Show that the wave function $\psi(x) = e^{ik_1x} + e^{ik_2x}$ is a pure carrier wave of frequency $\bar{k} = \frac{1}{2}(k_2 + k_1)$ modulated by a sinusoidal envelope of frequency $\hat{k} = \frac{1}{2}(k_2 - k_1)$. Assuming $\hbar\omega = \hbar^2k^2/2m$, determine the phase and group velocity, and associate each with one of the features above.

$k_1 = \bar{k} - \hat{k}$ $\psi(x) = e^{i(\bar{k}-\hat{k})x} + e^{i(\bar{k}+\hat{k})x} = (e^{-i\hat{k}x} + e^{i\hat{k}x})\, e^{i\bar{k}x}$

$k_2 = \bar{k} + \hat{k}$ $= \underbrace{2\cos(\hat{k}x)}_{\text{packet}}\, \underbrace{e^{i\bar{k}x}}_{\text{phase}}$ 6



phase carrier

2 $v_\phi = \frac{\bar{\omega}}{\bar{k}} = \frac{\hbar}{2m}\frac{k_1^2+k_2^2}{k_1+k_2} = \frac{\hbar\bar{k}}{2m}\left(1+\frac{\hat{k}^2}{\bar{k}^2}\right) \approx \frac{1}{2}v$ phase velocity.

2 $v_g = \frac{\hat{\omega}}{\hat{k}} = \frac{\hbar}{2m}\left(\frac{4\bar{k}\hat{k}}{2\hat{k}}\right) = \frac{\hbar\bar{k}}{m} = v$ group velocity.

[20 pts] 4. Sketch the three lowest energy eigenfunctions of the "staircase potential":

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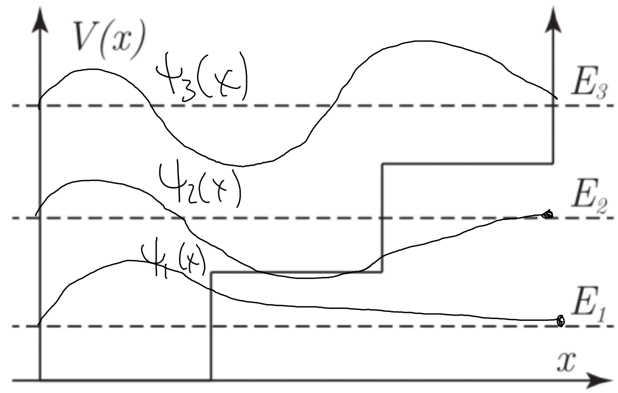

Calculate the normalized stationary state wave eigenfunctions $\psi_n(x)$ and energy eigenvalues E_n of the infinite square well potential $V(x) = 0$ if $0 < x < a$ and $V(x) = \infty$ otherwise. Sketch the first three wave functions and energy levels on a graph of the potential.

$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V(x)\psi = E\psi$ $\quad V(x)=0 \quad E=\frac{\hbar^2k^2}{2m}$

$-\psi'' = k^2\psi \quad \psi(x) = A\cos(kx) + B\sin(kx)$

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$\psi(0) = A = 0 \quad \psi(a) = B\sin(ka) = 0 \quad k_n a = n\pi$

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$\langle\psi|\psi\rangle = B^2\int_0^a \sin^2(k_n x) = B^2\frac{a}{2} = 1 \quad B = \sqrt{\frac{2}{a}}$

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thus $E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2ma^2} = \frac{h^2}{8ma^2}n^2$

$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(k_n x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$