EX2 - Solution

Monday, November 20, 2017 10:40

[8 pts] **1. a)** Show that if $N|n\rangle = n|n\rangle$ and $[N, a] = \alpha a$, then $a|n\rangle$ is another eigenstate of N with eigenvalue $n + \alpha$. Use this result to construct the energy spectrum and corresponding states of the harmonic oscillator, with Hamiltonian $H = \hbar\omega(N + \frac{1}{2})$ and ladder operators $a_\pm$, where $\alpha = \pm 1$.

$N(a|n\rangle) = (aN + [N,a])|n\rangle = an|n\rangle + \alpha a|n\rangle = (n+\alpha)a|n\rangle$

thus $a|n\rangle$ is an eigenvector of N with eigenvalue $n+\alpha$.

let $a_-|0\rangle = 0$ (lowest state) then $n=0$. From $\alpha = +1$, $n = 0, 1, 2, \ldots$

thus $E = \hbar\omega(n+\frac{1}{2})$, and $|n\rangle \propto a_+^n|0\rangle$.

[6 pts] **b)** Using $a_-\psi_0(\xi) = 0$ and the representation $a_- = \frac{1}{\sqrt{2}}(\partial_\xi + \xi)$ in the dimensionless variable ξ, calculate the ground state wave function $\psi_0(\xi)$.

$\frac{1}{\sqrt{2}}(\partial_\xi + \xi)\psi = 0 \qquad \frac{\partial\psi}{\partial\xi} = -\xi\psi \qquad \frac{d\psi}{\psi} = -\xi d\xi \qquad \ln \psi/N = -\frac{1}{2}\xi^2$

$\psi = Ne^{-\xi^2/2} \quad \left\{ \langle\psi|\psi\rangle = |N|^2 \int_{-\infty}^{\infty} d\xi\, e^{-\xi^2} = \sqrt{\pi} \quad N = (\pi)^{-1/4} \quad \text{OPTIONAL} \right\}$

[6 pts] **c)** Using $a_-|n\rangle = c_n|n-1\rangle$ with $||n-1\rangle|^2 = 1$ and $a_-^\dagger a_- = N$, obtain the normalization $a_-|n\rangle = \sqrt{n}|n-1\rangle$. Calculate the first 3×3 matrix elements of a_-.

$|a_-|n\rangle|^2 = \langle n|a_-^\dagger a_-|n\rangle = \langle n|N|n\rangle = n\langle n|n\rangle = n$

$= |c_n|n-1\rangle|^2 = |c_n|^2$ thus $c_n = \sqrt{n}$ or $a_-|n\rangle = \sqrt{n}|n-1\rangle$

where the phase of c_n was arbitrarily defined to 1.

[5 pts] **2. a)** Compare and contrast bound, scattering, and tunneling states.

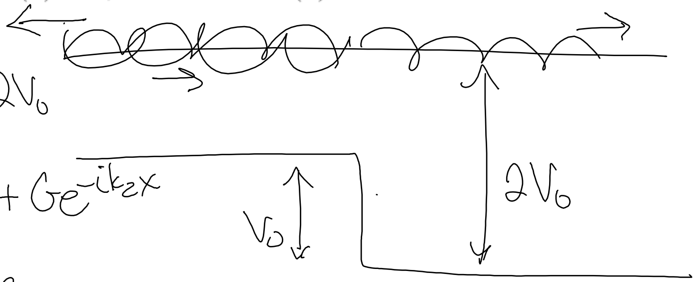
Bound: $T < 0$ as $x \to +\infty$ AND $-\infty$ (exponential decay both sides)
- discrete spectrum of normalizable states
- modes from external boundary conditions

Scattering: $T > 0$ as $x \to +\infty$ OR $-\infty$ OR both (plane wave one side)
- continuous spectrum, not normalizable

Scattering: $T>0$ as $x\to+\infty$ OR $-\infty$ OR both (plane wave one side)
- continuous spectrum, not normalizable
- external B.C.'s indicate incident flux from $-\infty, +\infty$.

Tunnelling: $T<0$ for a while, but $T>0$ going to $+\infty$ or $-\infty$
- represents probability of escape or decay per time.

[10 pts] **b)** Calculate the probability that a particle with kinetic energy $T = V_0$ will be reflected from the edge of a cliff of height V_0, i.e. find the coefficient of reflection R for a particle of energy $E = 2V_0$ incident from $x = -\infty$ in the potential $V(x) = V_0$ if $x < 0$ and $V(x) = 0$ if $x > 0$.



$T_1 = \frac{\hbar^2 k_1^2}{2m} = V_0 \quad T_2 = \frac{\hbar^2 k_2^2}{2m} = 2V_0$

$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x} \quad \psi_2 = Fe^{ik_2x} + Ge^{-ik_2x}$

ext. B.C.'s: A fixed and $G=0$.

int. B.C.'s: $\psi_1 = \psi_2|_0$: $(A + B = F) \times k_2$

$\psi_1' = \psi_2'|_0$: $k_1 A - k_1 B = k_2 F$

solve for B: $(k_2 - k_1)A + (k_2 + k_1)B = 0$

$$R = \left|\frac{B}{A}\right|^2 = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2 = \frac{\sqrt{1V_0} - \sqrt{2V_0}}{\sqrt{1V_0} + \sqrt{2V}} = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 = 2.9\%$$

[5 pts] **c)** What would happen classically in the same situation b)?

$R=0$, the particle would speed up going forward.

3. A two-state system with Hamiltonian $H = \begin{pmatrix} 0 & -B \\ -B & 0 \end{pmatrix}$ is in the initial state $\Psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The operator corresponding to the hypothetical observable 'hilarity' is $G = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

[5 pts] **a)** Calculate the energy eigenvalues and eigenstates.

$H - IE = \begin{pmatrix} -E & -B \\ -B & -E \end{pmatrix} = E^2 - B^2 = 0 \quad E = \pm B$

$E=+B$: $\begin{pmatrix} -B & -B \\ -B & -B \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \psi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad E=-B$: $\begin{pmatrix} B & -B \\ -B & B \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \psi_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$E=+B: \begin{pmatrix} -B & -B \\ -B & -B \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \psi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad E=-B: \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \psi_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

[5 pts] **b)** Calculate the time evolution of the state: $\Psi(t)$.

$$\psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\psi_+ + \frac{1}{\sqrt{2}}\psi_-$$

$$\psi(t) = \frac{1}{\sqrt{2}}\psi_+ e^{-iBt/\hbar} + \frac{1}{\sqrt{2}}\psi_- e^{+iBt/\hbar} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-iBt/\hbar} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{iBt/\hbar} = \begin{pmatrix} \cos Bt/\hbar \\ i\sin Bt/\hbar \end{pmatrix}$$

[5 pts] **c)** Find two possible values of 'hilarity' and the probability of measuring each at $t = 0$.

Note: $G = 3I - \frac{1}{B}H$ so $[G,H]=0$, thus

$$G\psi_+ = \left(3 - \tfrac{1}{B}(+B)\right)\psi_+ = 2\psi_+ \qquad G\psi_- = \left(3 - \tfrac{1}{B}(-B)\right)\psi_- = 4\psi_-$$

so $\psi_0 = \frac{1}{\sqrt{2}}\psi_+ + \frac{1}{\sqrt{2}}\psi_-$ means 50/50% chance of measuring either 4 or 2. Not that hilarious...

Long way: $G - \lambda I = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$ $\quad 3-\lambda = \pm 1$ $\qquad \lambda_\pm = 3 \pm 1 = 4, 2$

$\lambda=4: \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \lambda=2: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ note the swap between $E_\pm$ and $\lambda_\mp$ states.

[5 pts] **d)** Can both energy and 'hilarity' be measured at the same time? Compare and contrast *stationary* and *definite* states, using the state after measuring 'hilarity' as an example.

$$[H,G] = \begin{pmatrix} 0 & -B \\ -B & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & -B \\ -B & 0 \end{pmatrix} = \begin{pmatrix} -B & -3B \\ -3B & -B \end{pmatrix} - \begin{pmatrix} -B & -3B \\ -3B & -B \end{pmatrix} = 0$$

They can be simultaneously measured. Definite states of some obersable (eigenstates) will yield a particular measurement (the eigenvalue) with 100% certainty.

Stationary states are definite states of $\mathcal{H}$.

Hilarity definite states are also stationary because $[H,\sigma]=0$.

[3 pts] **4. a)** How is orthogonality used in Quantum Mechanics?

It is used to extract prob. amplitudes:

$$\langle\phi_n|\psi\rangle = \langle\phi_n|\left(c_1|\phi_1\rangle + c_2|\phi_2\rangle + \dots\right) = c_1\langle\phi_n|\phi_1\rangle^{0} + c_2\langle\phi_n|\phi_2\rangle^{0} + \dots c_n\langle\phi_n|\phi_n\rangle + \dots$$

so $c_n = \langle\phi_n|\psi\rangle / \langle\phi_n|\phi_n\rangle$.

[4 pts] **b)** Describe in detail how unitary operators are used in Quantum Mechanics. What is is the connection between orthogonal states and unitary operators?

They transform from one basis to another.

The columns of a unitary matrix are orthogonal. (it preserves orthogonality: lengths & angles).

[4 pts] **c)** Describe in detail how Hermitian operators are used in Quantum Mechanics. What is is the connection between orthogonal states and Hermitian operators?

They represent observables: eigenstates are definite states and eigenvalues are corresponding measurements.

The eigenstates are guaranteed orthogonal by the Sturm-Liouville theorem.

[5 pts] **b)** Using $\langle u_i|H|u_j\rangle$, show that if $H|u_i\rangle = |u_i\rangle\lambda_i$ and $H^\dagger = H$, then λ_i is real.

$H|u_i\rangle = |u_i\rangle\lambda_i \quad \lambda_i^*\langle u_i|u_j\rangle = \langle u_i|H|u_j\rangle = \langle u_i|u_j\rangle\lambda_j$

$\langle u_i|H^\dagger = \lambda_i^*\langle u_i| \quad$ if $i=j$ then $\langle u_i|u_i\rangle \neq 0$ so $\lambda_i = \lambda_i^*$

[4 pts] **e)** Apply the result from 1a) with $\alpha = 0$: if $N|n\rangle = n|n\rangle$ and $[N,a] = 0$ then $N|n'\rangle = n|n'\rangle$ where $|n'\rangle = a|n\rangle$, to explain the significance of commuting operators.

if $[N,a]=0$ then $a|n\rangle$ is another eigenvector of N with the same eigenvalue. Thus a preserves the eigenspaces of N, and a is block diagonal in the basis $|n\rangle$. Furthermore, a can be diagonalized, keepin the blocks of $N = \lambda_1 I \oplus \lambda_2 I \oplus \ldots$ diagonal.

Since N, a are simultaneously diagonalizable, they have the same definite states.