University of Kentucky, Physics 520
Homework #3, Rev. A, due Wednesday, 2017-09-20

0. Griffiths [2ed] Ch. 1 #9.

1. Waves on a string of linear mass density μ stretched horizontally with tension T.

a) Derive the wave equation. Hint: apply Newton's law, ignoring gravity, to a segment of string of infinitesimal length dx and mass dm.

b) Substitute the wave function $f(x, t) = Ae^{i(kx-\omega t)}$ into the wave equation to derive the dispersion relation $\omega(k)$. Use this to calculate the phase velocity of waves on the string.

c) A string of length L is fixed at one end. The other end is tied to a massless ring, freely sliding up and down a rod. Use boundary conditions to find the spectrum of allowed frequencies.

d) In analogy with AC electrical impedance $Z = V/I = R + i\omega L + 1/i\omega C$, the mechanical impedance of an oscillating system (mass M, spring constant K, drag coefficient B) is $Z = F/v = B + i\omega M + K/i\omega$. Calculate the characteristic impedance of the string, $Z = Tf'/\dot{f}$ (ratio of vertical force to velocity) in terms of T and μ. Show the power transferred along the wave is $P = Z\dot{f}^2$.

e) Two strings of density $\mu_{1,2}$ and tension $T_{1,2}$ are joined by a ring on a rod to support the difference in tension. Justify the boundary conditions $\Delta f = 0$ and $\Delta(Tf') = \Delta(\pm Z\dot{f}) = 0$. Hint: apply Newton's law to vertical forces on the ring to obtain the second condition.

f) An incident wave $A_I e^{i(k_1 x-\omega t)}$ from the left is partially reflected at the ring due to the change in impedance. The reflected wave $A_R e^{i(-k_1 x-\omega t)}$ is superimposed on the incident wave, and the forward transmitted wave is $A_T e^{i(k_2 x-\omega t)}$. Apply the two boundary conditions to obtain A_R/A_I and A_T/A_I. Calculate the coefficients of reflected power $R = Z_1 A_R^2/Z_1 A_I^2$ and transmitted power $T = Z_2 A_T^2/Z_1 A_I^2$ in terms of $Z_{1,2}$. Show they add up to 100%.

g) [bonus] Repeat for a ring of impedance Z (mass M, spring constant K, damping B).

2. As **gravity waves** propagate along the interface between a liquid and a gas, particles in the liquid follow elliptical trajectories with an amplitude which decays exponentially with depth. If the velocity field $\vec{v}(x, z)$ is irrotational $\nabla \times \vec{v} = 0$, we can represent it as the gradient of a scalar flow potential $\vec{v} = -\nabla\phi(x, z)$. The flow of an incompressible fluid $\nabla \cdot \vec{v} = 0$, satisfies the Laplace equation $\nabla^2\phi = 0$. Let the gas-liquid interface have height $z = \eta(x, t)$ above the equilibrium level $z = 0$. In addition to gravity, surface tension γ, exerts pressure $P = \gamma\nabla^2\eta$ on the liquid, causing the wave to propagate.

a) Show that the function $\phi(x, z, t) = a\cosh(k(z + h))\sin(kx - \omega t)$ is a solution of $\nabla^2\phi = 0$. Plot equipotentials of ϕ at $t = 0$, with arrows showing the direction of $\vec{v}$.

b) Show this solution satisfies the boundary condition $v_z(-h) = 0$ at the bottom of the liquid.

c) The boundary condition on the top surface is $v_z = \dot{\eta}$, evaluated at $z = 0$ (approximately at the boundary). Show that ϕ satisfies the boundary condition $\eta(x, t) = A\cos(kx - \omega t)$.

d) Integrating Newton's law over z leads to Bernoulli's law $\partial_t\phi = g\eta - \frac{\gamma}{\rho}\partial_x^2\eta$, where ρ is the mass density of the liquid. Substitute ϕ and η into Bernoulli's law to obtain the dispersion relation $\omega^2 = (gk + \frac{\gamma}{\rho}k^3)\tanh(kh)$. Plot $\omega(k)$, $v_\phi(k) = \omega/k$, and $v_g(k) = d\omega/dk$.

e) Calculate the wavelength λ_c below which waves are dominated by surface tension ($\gamma = 72.8$ mN/m and $\rho = 1.00$ g/cm^3 for water). What is the dispersion relation in this limit?

f) Approximate $\phi(x, z, t)$ and $\omega(k)$ in the deep water limit, where $kh >> 1$. Do individual crests move forward or backward within the wave packet?

g) Approximate $\omega(k)$ in the shallow water limit, and show that all frequencies have the same velocity. What is the speed of a tsunami ($\lambda \approx 100$ km) in 10 km deep [shallow!] ocean waters? How long does it take one wavelength to pass?