University of Kentucky, Physics 520
Homework #7, Rev. A, due Monday, 2017-10-23

0. Griffiths [2ed] Ch. 2 #15, #16, #17, #42.

1. Parity harmonics — stationary states of the harmonic oscillator alternate between even and odd functions of x with increasing energy. We will explicitly solve the even or odd states by restricting the domain to $x > 0$ and applying the boundary condition $\psi'(0) = 0$ or $\psi(0) = 0$, respectively. Doing so leads to solutions in terms of the associated Laguerre polynomials instead of the standard Hermite polynomials.

a) Show that the wave function $\psi(\xi) = \xi^p e^{-\xi^2/2} L(\xi^2)$ satisfies the boundary condition for either even or odd functions of ξ with $p = 0$ or 1, respectively, for any function $L(u)$. Note that the exponential captures the asymptotic dependence of $\psi(\xi)$.

b) Substitute $\psi(\xi)$ into Griffiths Eq. 2.72 [to save effort, substitute $h = \xi^p L(\xi^2)$ in Eq. 2.78], and change variables from ξ to $u = \xi^2$ to obtain the associated Laguerre differential equation $uL'' + (p - \frac{1}{2} + 1 - u)L' + kL = 0$ [wikipedia.org/Laguerre_polynomials]. *Hint: use the chain rule to calculate* $h'(\xi)$. What is the relation between E and k?

c) Solve the associated Laguerre equation using the Frobenius method: substitute the Taylor series $L(u) = \sum_{j=0}^{\infty} a_j u^j$, and solve the recurrence relation for a_j.

d) Calculate the energy levels for both $p = 0$ and $p = 1$ by setting $k = 0, 1, 2, \ldots$ to truncate the series so that $\psi(\pm\infty) \to 0$. Compare with the levels $E_n = \hbar\omega(n + \frac{1}{2})$, where $n = 2k + p$.

e) Use the recursion relation to calculate the first three solutions $L_k(u)$, where $k = 0, 1, 2$, for both $p = 0$ and $p = 1$, using the normalization $L_k(0) = \binom{k+\alpha}{k} = \frac{(1+\alpha)(2+\alpha)\cdots(k+\alpha)}{1 \cdot 2 \cdots k}$, where $\alpha = p - \frac{1}{2}$. Compare your solutions with the associated Laguerre polynomials $L_k^{p-\frac{1}{2}}(u)$. Note that the first three are $L_0^{(\alpha)}(u) = 1$, $L_1^{(\alpha)}(u) = (1 + \alpha - u)$, and $L_2^{(\alpha)}(u) = \frac{(1+\alpha)(2+\alpha)}{1 \cdot 2} - (2 + \alpha)u + \frac{1}{2}u^2$.

f) **[bonus]** Normalize the wave functions $\psi_k^p(\xi)$. Compare these solutions to the standard wave-functions, Griffiths Eq. 2.85, to obtain the following relations between Hermite and associated Laguerre polynomials, Eqs. 22.5.40 and 22.5.41 of [Abramowitz and Stegun, p. 779]

$$H_{2k}(x) = (-1)^k 2^{2k} k! L_k^{-\frac{1}{2}}(x^2) \quad (1)$$

$$H_{2k+1}(x) = (-1)^k 2^{2k+1} k! x L_k^{\frac{1}{2}}(x^2). \quad (2)$$

We will encounter associated Laguerre polynomials in the solution of other potentials such as the 2d and 3d harmonic oscillators and the hydrogen atom.