University of Kentucky, Physics 520
Homework #10, Rev. A, due Wednesday, 2017-11-15

0. Griffiths [2ed] App. A #19, #22, #25.

1. The 2nd order linear **Sturm-Liouville** differential operator

$$L[y(x)] \equiv \frac{1}{w(x)} \left[\frac{d}{dx} p(x) \frac{d}{dx} - q(x) \right] y(x) \tag{1}$$

is self-adjoint, $L^\dagger = L$, with respect to the inner product

$$\langle y_1 | y_2 \rangle \equiv \int_a^b w(x) dx \; y_1(x)^* \, y_2(x) \tag{2}$$

if we impose the boundary conditions $y(a)w(a) = y(b)w(b) = 0$. Thus L has real eigenvalues λ_i and a complete set of orthogonal eigenfunctions $u_i(x) = \langle x|u_i\rangle$. In particular, $L|u_i\rangle = \lambda_i|u_i\rangle$, with $\langle u_i|u_j\rangle = \delta_{ij}$ [orthonormality], and $\sum_i |u_i\rangle\langle u_i| = I$ [completeness]. This means that any smooth function $|f\rangle$ can be expanded in the basis $|u_i\rangle$ as $|f\rangle = \sum_i |u_i\rangle f_i$ or $f(x) = \sum_i u_i(x) f_i$, with components $f_i = \langle u_i|f\rangle = \int_a^b w(x)dx \; u_i(x)^* f(x)$.

a) Show that L is self-adjoint or Hermitian. *Hint:* use the definition $\langle f|H^\dagger g\rangle \equiv \langle Hf|g\rangle$ to show that the derivative operator $\frac{d}{dx}$ is anti-Hermitian and apply it to the composition of operators in L.

b) Given eigenfunctions $L|u_i\rangle = \lambda_i|u_i\rangle$, show that $\lambda_i \in \mathbb{R}$ and that $\langle u_i|u_j\rangle = 0$ if $\lambda_i \neq \lambda_j$. Note that it is much harder to prove completeness. Operate L on the expansion of $|f\rangle$ in the basis $|u_i\rangle$ to show that its *spectral decomposition* is $L = \sum_i \lambda_i |u_i\rangle\langle u_i|$. What is the decomposition of the identity operator $I|f\rangle = |f\rangle$ in the same orthogonal basis $|u_i\rangle$?

2. a) The **Legendre polynomials** $P_n(\cos\theta)$, used for spherically symmetric potentials, are eigenfunctions of the operator $L = \frac{d^2}{d\theta^2} + \cot\theta \frac{d}{d\theta}$. Show that this is a Sturm-Liouville system on the domain $0 < \theta < \pi$, with $w(\theta) = \sin\theta$, $p(\theta) = \sin\theta$, and $q(\theta) = 0$. Change variables to $x = \cos\theta$ and calculate the new functions $w(x)$, $p(x)$, $q(x)$ and domain $a < x < b$. Note the sign change!

b) Show that $\langle x^m|x^n\rangle = \frac{2}{m+n+1}$ if $m+n$ is even and, 0 if $m+n$ is odd. Apply the Gram-Schmidt procedure to the basis functions 1, x, and x^2 to obtain the first three Legendre polynomials $P_\ell(x)$, and find their eigenvalues λ_ℓ.

3. Compile a chart of w, p, q, λ_i for each of the following **orthogonal functions** $\phi_i(x)$:

i)	Cylindrical harmonics	$e^{im\phi}$	on	$0 < \phi < 2\pi$.
ii)	Associated Legendre functions	$P_l^{\lvert m\rvert}(x)$	on	$-1 < x < 1$.
iii)	Fourier series	$\sin(k_n x)$	on	$0 < x < b$.
iv)	Bessel functions	$J_m(k_n x)$	on	$0 < x < b$.
v)	Spherical Bessel functions	$j_l(k_n x)$	on	$0 < x < b$.
vi)	Hermite polynomials	$H_n(x)$	on	$-\infty < x < \infty$.
vii)	Associated Laguere polynomials	$L_n^{(\alpha)}(x)$	on	$0 < x < \infty$.

4. Simultaneously diagonalize the matrices $A = \begin{pmatrix} 9 & -2 & -6 \\ -2 & 9 & -6 \\ -6 & -6 & -7 \end{pmatrix}$ and $B = \begin{pmatrix} 54 & 10 & -3 \\ 10 & -45 & 30 \\ -3 & 30 & 46 \end{pmatrix}$.

Are the eigenvectors orthogonal? *Hint:* Octave or Mathematica is your friend!