University of Kentucky, Physics 520
Homework #11, Rev. B, due Monday, 2017-12-04

0. Griffiths [2ed] Ch. 4 #2, #3.

1. We will see that **central potentials** have effectively the same Schrödinger equation in all dimensions. We will use this fact to solve the 2-d and 3-d harmonic oscillators using the solution of the 1-d harmonic oscillator from H07.

a) Calculate the Laplacian in cylindrical and spherical coordinates using

$$\nabla^2 = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial q^i} \frac{h_j h_k}{h_i} \frac{\partial}{\partial q^i} \qquad [i, j, k \text{ cyclic}].$$

b) Substitute eigenvalues m^2 and $\ell(\ell+1)$ for the angular parts and derive the formulas

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} = \frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial \rho^2}\sqrt{\rho} - \frac{(m-\frac{1}{2})(m+\frac{1}{2})}{\rho^2}$$

in 2-d cylindrical coordinates (ignoring z) and

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{\ell(\ell+1)}{r^2}$$

in spherical coordinates.

c) Substitute $\psi(\rho,\phi) = \frac{u(\rho)}{\sqrt{\rho}}e^{im\phi}$ and $\psi(r,\theta,\phi) = \frac{u(r)}{r}P_\ell^{|m|}(\cos\theta)e^{im\phi}$ into the 2-d and 3-d Schrödinger equations $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$ for central potentials $V(\rho)$ and $V(r)$ to obtain the 1-d Schrödinger equation for $u(\rho)$ or $u(r)$ with the same potential plus a centrifugal term involving m or ℓ. The factors of $\sqrt{\rho}$ and r account for the wavefunction spreading out in space.

d) Comparing the equations for the 2-d and 3-d harmonic oscillator, $V(\rho) = \frac{1}{2}m\omega^2\rho^2$ and $V(r) = \frac{1}{2}m\omega^2 r^2$, with the 1-d Hamiltonian and its parity solutions in H07,

$$\mathcal{H}_x = -\frac{\hbar^2}{2m}\left[\frac{d^2}{dx^2} - \frac{(p-1)(p)}{x^2}\right] + \frac{1}{2}m\omega^2 x^2, \qquad \psi(\xi) = \frac{\Gamma(p+\frac{1}{2}+k)}{k!}\xi^p e^{-\xi^2/2}L_k^{(p-\frac{1}{2})}(\xi^2), \quad n = 2k+p,$$

where $x, \rho, r = \sqrt{\hbar/m\omega}\,\xi$, derive the radial functions $u(\rho)$ and $u(r)$, respectively, and calculate the energy levels of the 2-d and 3-d harmonic oscillators. The centrifugal term $(p-1)(p)/x^2$ is zero since $p = 0$ (even) or 1 (odd), but it is included to help determine the relation between p, m, and ℓ.

e) [bonus] The 2-d harmonic oscillator Hamiltonian separates into the sum of two 1-d Hamiltonians $\mathcal{H} = \mathcal{H}_x + \mathcal{H}_y$, with the solution $\psi_{n_x n_y}(x,y) = \psi_{n_x}\psi_{n_y}$ for independent quantum numbers n_x and n_y. Compare the wave functions for $n_x + n_y \le 2$ with the wave functions from part d) for $n, |m| \le 2$ and show the two sets are linear combinations of each other.

f) The same symmetry between dimensions holds for all central potentials, including the free particle. Use part c) to show that the 1-d, 2-d, and 3-d free particle solutions can all be written in terms of $J_\nu(x)$, where ν is a function of p, m, or l.