L05-Anatomy of a Wave

Sunday, September 4, 2016 09:43

* We will be studying "wave mechanics", in Ch2, Schrödinger's program of quantizing energy states in terms of standing waves.
  - Before studying quantum wave functions, we should understand the features of classical waves.
  - This will lead to a natural motivation of the T.D.S.E.

* What is a wave?

  - A wave is a collective mode of oscillation that can transfer energy and momentum from one location to another without the corresponding transfer of material (mass)

  - It is characterized by properties of the underlying medium: tension, density → velocity (dispersion relation), impedance (energy storage)
    And also by properties of the mode of oscillation: wavelength, frequency (spectrum), amplitude, polarization
    Compare: particles have static moments (mass, centre of mass, inertia) and degrees of freedom: position $\vec{x}(t)$ → velocity → energy, momentum

  - Energy is transferred as it oscillates between different forms in the medium, for example potential [tension] & kinetic [inertia].
    Compare: particles transfer pure kinetic energy by motion

  - The state is described by a wave function, which evolves according to a PDE wave equation
    Compare: particles have a trajectory following an ODE equation of motion

  - Mechanical waves are composed of particles (usually bound)
    0d { single particle: oscillator $m\ddot{x}+b\dot{x}+kx=f(t) \rightarrow x=A\cos(\omega t+\phi)$
    { multiple particles: collective mode: $\mathbb{M}\ddot{\mathbb{X}}+\mathbb{K}\mathbb{X}=0 \rightarrow \mathbb{X}=\sum_i \eta_i \cos(\omega_i t+\phi_i)$
    1,2,3d { continuous medium $\boxed{x_i \rightarrow f(\vec{x})}$ "$\vec{x}$" is now a particle index!
    Other waves [E&M, quantum, gravitational] travel through a medium of underlying fields, not particles.

* What is the difference between a classical particle and wave?

PROPERTY PARTICLE WAVE

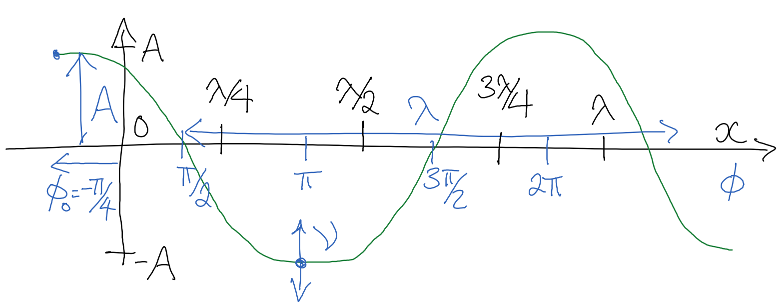
* What is the difference between a classical particle and wave?

PROPERTY	PARTICLE	WAVE
material properties:	mass "m", inertia "I"	velocity "v", impedance "Z"
degrees of freedom:	$\vec{r}(t)$ trajectory (vector)	$\Phi(\vec{r},t)$ wave function (field)
(Fourier space)		Amplitude$(\vec{k},\omega)$, polarization
dynamics:	$\vec{F}=m\vec{a}$ [ODE]	$(\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2})\Phi(\vec{r},t)=0$ [PDE]
dispersion	$E=p^2/2m$ $v=dE/dp$	$v=\lambda\nu=\omega/k$ or $d\omega/dk$
conservation	$(E,\vec{p})=$const	$\nabla\cdot(\vec{S},\overleftrightarrow{T})+\partial_t(u,\vec{p})=0$
quantization	mass (Brownian motion), charge	modes (standing waves)
sociality	individual D.o.F.	collective motion of particles
interactions	collision, forces, detection	reflection/refraction/diffraction

* Examples of waves:
  1) Material waves, phase: solid, liquid, gas
     dimension: 1-d (tension), 2-d (surface tension, gravity), 3-d (body)
     Examples: string: tension T, linear mass density μ
     gravity: gravity g, (mass density cancels!)
     acoustic: pressure P, density ρ.
     seismic: stress/strain moduli, density.
  2) Electromagnetic waves, 1,2,3-d wave guides.
     E, B fields, permittivity ε (tension), permeability μ (inertia)
  3) Gravitational waves
     metric tensor field, energy/curvature G, ?
  4) Quantum mechanical "phase" waves
     probability amplitude, kinematic dispersion $H=\frac{p^2}{2m}+V$

* Properties of a one-dimensional wave function

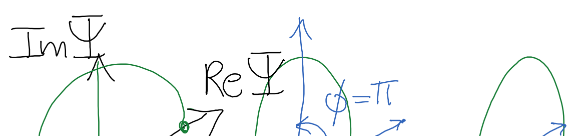
$$\Psi(x,t)=A_0\cos(\underbrace{kx-\omega t-\phi_0}_{\text{phase }\phi})$$



- It's best to convert everything to radians and think in terms of phase.

$\omega=2\pi\nu=\frac{2\pi}{T}$ $k=\frac{2\pi}{\lambda}$ "wave number" $v=\lambda\nu=\frac{\omega}{k}$ "dispersion"

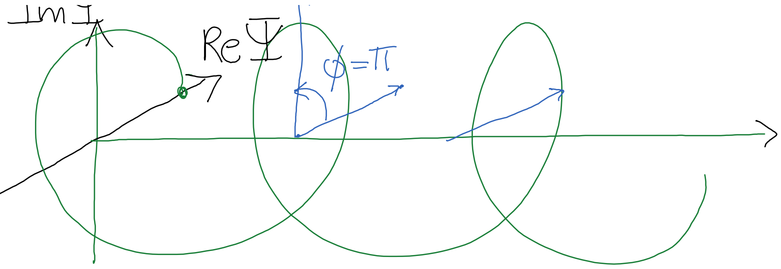
* Complex amplitudes



* Complex amplitudes

$e^{i\phi} = \cos\phi + i\sin\phi =$ "$\mathrm{cis}(\phi)$"

$\Psi(x,t) = \mathrm{Re}\left[A_0 e^{-i\phi_0} \cdot e^{i(kx-\omega t)}\right]$



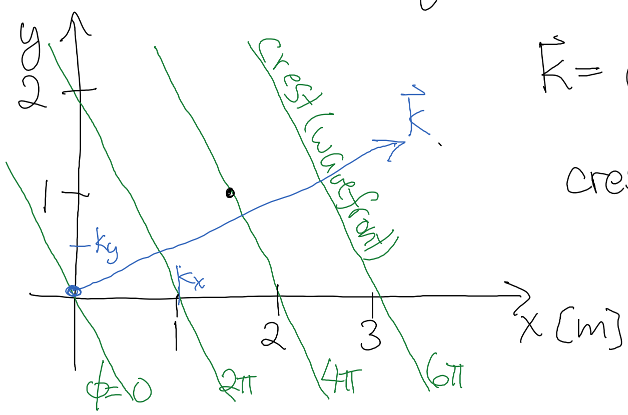
Advantage: everything factors! (separation of variables) $e^{ikx} \cdot e^{-i\omega t}$

- complex amplitude includes phase (phasors)
- Imaginary part captures the velocity (1st order equation)
- Real exponenent represents attenuation $e^{-\kappa x}$

* Exponentials are "eigenfunctions" of derivative operators.

$$\underbrace{\boxed{\frac{\partial}{\partial x}}}_{\text{operator}} e^{ikx} = \underbrace{ik}_{\text{eigenvalue}} e^{ikx} \qquad \underbrace{\boxed{\frac{\partial}{\partial t}}}_{\text{operator}} e^{-i\omega t} = \underbrace{-i\omega}_{\text{eigenvalue}} \underbrace{e^{-i\omega t}}_{\text{eigenfunction}}$$

* Plane waves in higher dimension (2 or 3-d) $\boxed{\Psi(x,y) = e^{ik_x x} e^{ik_y y} = e^{i\vec{k}\cdot\vec{r}}}$



$\vec{k} = (k_x, k_y) = (2,1) \cdot \pi \ \frac{\text{rad}}{\text{m}}$ $\quad \omega = 2\pi \ \frac{\text{rad}}{\text{s}}$

crest: $\vec{k}\cdot\vec{r} = k_x x + k_y y = \phi = \text{const}$

$v_n = \dfrac{\omega}{\vec{k}\cdot\hat{n}} \quad v_x = 1 \ \text{m/s} \quad v_y = 2 \ \text{m/s}$

$$\underbrace{\boxed{\nabla}}_{\text{operator}} e^{i\vec{k}\cdot\vec{r}} = \underbrace{i\vec{k}}_{\text{eigenvalue}} e^{i\vec{k}\cdot\vec{r}} \ \text{eigenfunction}$$