# L06-Wave packets and group velocity

Friday, August 25, 2017 08:33

* Wave particle duality:

- we've compared and contrasted waves and particles
- quantization of radiation and matter was achieved by unifying the concepts of particles and waves

• Planck: (2nd quantization) $E = \hbar\nu$ EM waves discretized as "photons"
• de Broglie: (1st quantization) $p = \hbar k$ matter energy quantized as "modes"

* What is the connection between particles and waves?

$\boxed{\text{Everything propagates as a wave, and is detected as a particle.}}$

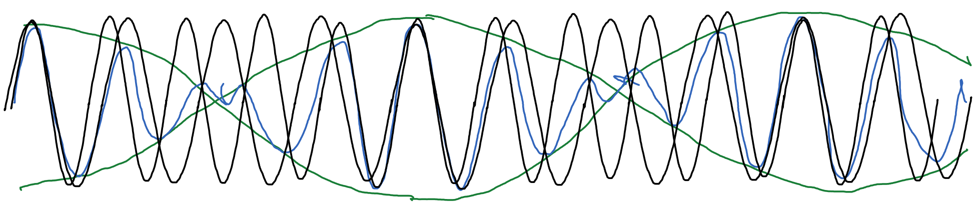
- The "position" of a wave only makes sense for localized waves (wave packets vs. pure frequency waves).

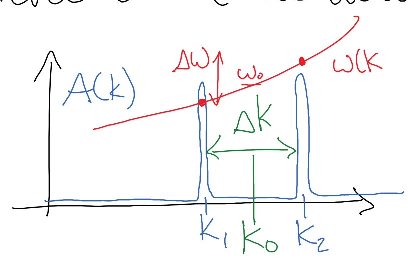
- The "medium" of a quantum mechanical "phase wave" is probability amplitude: $P(x) = |\psi(x)|^2 = \psi^*(x)\cdot\psi(x)$
  $\int_a^b |\psi(x)|^2 dx$ is the probability of finding the particle at $a \le x \le b$

- Compare E&M wave: energy density $u = \frac{1}{2}\varepsilon_0 |E(\vec{r})|^2$ (quadratic)

* Simplest example of a wave packet: beats

- two instruments slightly out of tune produce a beating tone due to alternating positive and negative interference of the two waves






$$f(x,t) = A e^{i(k_1 x - \omega_1 t)} + A e^{i(k_2 x - \omega_2 t)} = A(e^{i\phi_1} + e^{i\phi_2})$$

$$\phi_i \equiv k_i x - \omega_i t \quad i = 1,2$$

$$= A e^{i\bar{\phi}}(e^{i\tilde{\phi}} + e^{-i\tilde{\phi}}) = A e^{i\bar{\phi}} \cdot 2\cos\tilde{\phi}$$

$$\phi_{1,2} \equiv \bar{\phi} \pm \tilde{\phi} \qquad \bar{\phi} = \tfrac{1}{2}(\phi_1 + \phi_2) \quad \tilde{\phi} = \tfrac{1}{2}(\phi_1 - \phi_2)$$

$$= A \underbrace{e^{i(\bar{k}x-\bar{\omega}t)}}_{\text{phase/carrier wave}} \cdot \underbrace{2\cos(\Delta k \cdot x - \Delta\omega t)}_{\text{group/packet/beats/modulation.}}$$

- the "carrier wave" (average frequency) travels at the average "phase velocity" of the two waves $\quad v_\phi = \bar{\omega}/\bar{k}$

- the "envelope/modulation packet" travels at the "group velocity" (slope of dispersion relation) $\quad v_g = \Delta\omega/\Delta k \to d\omega/dk$

- the envelope gets wider as the beating frequencies get closer
  $\Delta k \cdot \Delta x \approx \pi \quad \Rightarrow \quad \Delta x \approx \pi/\Delta k$

* Generalization: Fourier Series – synthesis of $f(x)$ with period $\lambda_1 = \frac{2\pi}{k_1}$

$$f(x) = \sum_n A_n \cos(k_n x) + B_n \sin(k_n x) \qquad \sum_{n=-\infty}^{\infty} C_n e^{ik_n x} \qquad k_n = n \cdot k_1$$

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \frac{2}{\lambda_1}\int_0^{\lambda_1} f(x) \begin{bmatrix} \cos(k_x) \\ \sin(k_x) \end{bmatrix} dx \qquad \text{or} \qquad C_n = \frac{1}{\lambda_1}\int_0^{\lambda_1} f(x)\, e^{-ik_n x}\, dx$$

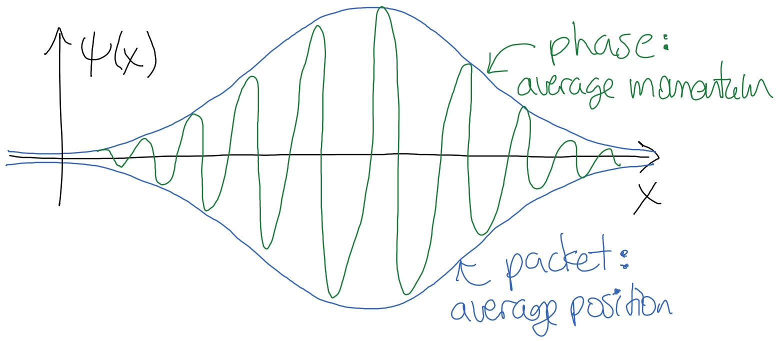
Continuous limit: $k_1 \to dk$, $\sum_n A_n \to \int dk\, A(k)$ is the Fourier transform

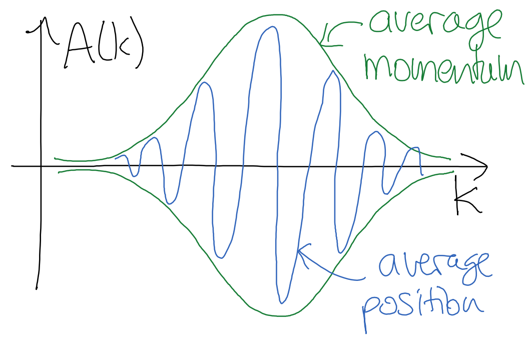
$$f(x) = \frac{1}{\sqrt{2\pi}}\int dk\, A(k)\, e^{ikx} \quad \text{and it's inverse:} \quad A(k) = \frac{1}{\sqrt{2\pi}}\int dx\, f(x)\, e^{-ikx}$$

note: sometimes the $\frac{1}{\sqrt{2\pi}}$'s are combined $\tilde{f}(k) = \frac{1}{2\pi}\int dx\, f(x)\, e^{-ikx}$

* explore PhET applet https://phet.colorado.edu/en/simulation/fourier
- $k_1 = \frac{2\pi}{\lambda_1}$ fundamental frequency, repetition period $\lambda_1$
  frequency of the wave packets [discrete only]
  $k_1 = dk \to 0$ for Fourier transform
- $A_n$ peaked at $k_0 = nk_1$ carrier (phase) frequency
  apparent frequency of the wave
- $\Delta k$ = frequency bandwidth (modulation)
  sharpness of packet in position space
  highest frequency of wave packet

* A wave packet is smeared over both position AND momentum.
It is the superposition of a whole spectrum of pure frequency waves.




$$\Psi(x) = \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$$

where

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ikx}\cdot \Psi(x)$$

" Fourier transformation" (or decomposition)

* A direct consequence of Fourier decomposition is the "Heisenberg Uncertainty Principle"

$$\Delta k \cdot \Delta x \geq \tfrac{1}{2} \quad \Rightarrow \quad \Delta p \cdot \Delta x \geq \hbar/2$$
$$\Delta \omega \cdot \Delta t \geq \tfrac{1}{2} \quad \Rightarrow \quad \Delta E \cdot \Delta t \geq \hbar/2$$

The position and momentum of a particle are both intertwined in the wave function and cannot be specified separately.
"Bohr's Complementarity Principle" (wave particle duality)

The minimum uncertainty wave $\Delta k \cdot \Delta x = \hbar/2$ is a Gaussian wave packet

$$\Psi(x) = \int dk\, A(k)\, e^{ikx} \qquad A(k) = N e^{\frac{1}{4}\left(\frac{k-k_0}{\Delta k}\right)^2}$$