L07-Heisenberg Uncertainty Principle: Fourier Transform

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* family of Gaussian integrals: $\boxed{I_n \equiv \int_0^\infty dx\, x^n e^{-\alpha x^2}}$

$$I_1 = \int_0^\infty x e^{-\alpha x^2} dx = \frac{1}{2\alpha}\int_0^\infty e^{-u}\, du = \frac{1}{2\alpha}$$

$u = \alpha x^2$
$du = \alpha \cdot 2x dx$

$$I_0^2 = \int_0^\infty\int_0^\infty dx\,dy\; e^{-\alpha(x^2+y^2)} = \int_0^\infty s\,ds\; e^{-\alpha s^2}\int_0^{\pi/2} d\phi = \frac{\pi}{2} I_1 = \frac{\pi}{4\alpha}$$

$$I_{n+2} = \int_0^\infty x^{n+2} dx\, e^{-\alpha x^2} = -\frac{\partial}{\partial\alpha} I_n = \frac{n+1}{2\alpha} I_n$$

$I_2 = -\frac{\partial}{\partial\alpha}\frac{1}{2}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$

n	0	1	2	3	4	5	6
$\frac{1}{2}(n+1)$	1/2	1	3/2	2	5/2	3	7/2
$2I_n$	$\sqrt{\frac{\pi}{\alpha}}$	$\frac{1}{\alpha}$	$\frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$	$\frac{1}{\alpha^2}$	$\frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$	$\frac{2}{\alpha^3}$	$\frac{15}{8}\sqrt{\frac{\pi}{\alpha^7}}$

$$I_{2k} = \left(\frac{2k-1}{2\alpha}\right)\left(\frac{2k-3}{2\alpha}\right)\cdots\left(\frac{1}{2\alpha}\right) I_0 = \frac{(2k-1)!!}{(2\alpha)^k}\cdot\frac{1}{2}\sqrt{\frac{\pi}{\alpha}} = \boxed{\frac{(2k)!}{k!}\frac{1}{(2\sqrt{\alpha})^{2k+1}}\sqrt{\pi}} = \frac{\Gamma(k+\frac{1}{2})}{2\;\alpha^{k+\frac{1}{2}}}$$

$\downarrow n=2k$

$$I_{2k+1} = \left(\frac{2k}{2\alpha}\right)\left(\frac{2k-2}{2\alpha}\right)\cdots\left(\frac{2}{2\alpha}\right) I_1 = \boxed{\frac{k!}{\alpha^k}\frac{1}{2\alpha}} = \frac{\Gamma(k+1)}{2\alpha^{k+1}} \xrightarrow{n=2k+1} I_n = \frac{\Gamma(\frac{n+1}{2})}{2\,\alpha^{\frac{n+1}{2}}}$$

$$\Gamma(t) \equiv \int_0^\infty u^{t-1} e^{-u} du$$

let $u = \alpha x^2$ $\quad du = \alpha\cdot 2x dx$

$$= \int_0^\infty (\alpha x^2)^{t-1} e^{-\alpha x^2}\cdot\alpha\cdot 2x dx$$

$$= 2\alpha^t \int_0^\infty x^{2t-1} e^{-\alpha x^2} dx$$

$$= 2\alpha^t I_{2t-1}$$

$n = 2t-1$
$t = \frac{1}{2}(n+1)$

so $\boxed{2I_n = \frac{\Gamma(\frac{1}{2}(n+1))}{\alpha^{\frac{1}{2}(n+1)}}}$

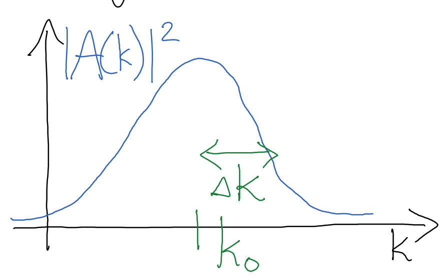
$$\Gamma(t+1) = \int_0^\infty u^t e^{-u} du$$
$$= \int_0^\infty u^t\, d(e^{-u})$$
$$= \cancel{-u^t e^{-u}\Big|_0^\infty} + \int_0^\infty e^{-u} du^t$$
$$= t\int_0^\infty u^{t-1} e^{-u} du$$

$$\left.\begin{array}{l} \Gamma(t+1) = t\,\Gamma(t) \\ \Gamma(n+1) = n! \\ \Gamma(\frac{1}{2}) = \sqrt{\pi} \end{array}\right\}\text{properties of } \Gamma(t)$$

* Moments of the Gaussian wave packet (following Gasiorowicz 2-6)

$$A(k) = N e^{-\frac{1}{4}\left(\frac{k-k_0}{\Delta k}\right)^2}$$

$|A(k)|^2$ normal distribution



Normalization: $\int |\Psi(x)|^2 dx = \int |A(k)|^2 dk = 1$

$$\int_{-\infty}^{\infty} N^2 e^{-\frac{1}{2}\left(\frac{k-k_0}{\Delta k}\right)^2} dk$$

$x = \frac{k-k_0}{\Delta k}$ $\quad dk = \Delta k\, dx$

$$= N^2\Delta k\int e^{-x^2/2} dx = N^2\Delta k\cdot\sqrt{2\pi} = 1$$

so $N = \frac{1}{\sqrt{\sqrt{2\pi}\,\Delta k}}$

$-\infty$

$$= N^2 \Delta k \int_{-\infty}^{\infty} e^{-\chi^2/2} d\chi = N^2 \Delta k \cdot \sqrt{2\pi} = 1 \quad \text{so} \quad N = \frac{1}{\sqrt{\sqrt{2\pi}\,\Delta k}}$$

Mean: $$\langle k \rangle \equiv \int_{-\infty}^{\infty} N^2 e^{-\frac{1}{2}\left(\frac{k-k_0}{\Delta k}\right)^2} k\,dk = N^2 \Delta k \int_{-\infty}^{\infty} e^{-\chi^2/2} \big(\overset{\text{odd}}{\cancel{\Delta k \chi}} + k_0\big) d\chi = k_0$$

RMS: $$\langle k^2 \rangle \equiv \int_{-\infty}^{\infty} N^2 e^{-\frac{1}{2}\left(\frac{k-k_0}{\Delta k}\right)^2} k^2 dk = N^2 \Delta k \int_{-\infty}^{\infty} e^{-\chi^2/2} \underbrace{(\Delta k \chi + k_0)^2}_{\Delta k^2 \chi^2 + \cancel{2\Delta k \chi k_0} + k_0^2} d\chi$$

$$= N^2 \Delta k^3 \int_{-\infty}^{\infty} e^{-\chi^2/2} \chi^2 d\chi + k_0^2 = \Delta k^2 + \langle k \rangle^2 \quad \text{so} \quad \sigma_k = \Delta k$$

* Fourier Transform: to obtain the wave function $\psi(x)$

Note the factor $\frac{1}{\sqrt{2\pi}}$ ensures that $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} |A(k)|^2 dk$

$$\psi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx} = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{-\frac{1}{4}\chi^2 + i(\Delta k \chi + k_0)x}$$

$$= \frac{N}{\sqrt{2\pi}} \Delta k \underbrace{\int_{-\infty}^{\infty} du\, e^{-\frac{1}{4}u^2}}_{\sqrt{4\pi}} e^{-\Delta k^2 x^2} e^{ik_0 x}$$

$A(k) = N e^{-\frac{1}{4}\chi^2}$

$k = \Delta k \cdot \chi + k_0$

• complete the square:

$-\frac{1}{4}\chi^2 + i \Delta k x \chi = -\frac{1}{4}u^2 - \Delta k^2 x^2$

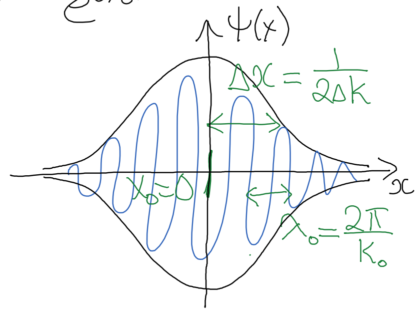
where $u = \chi - 2i\Delta k x$

$dk = \Delta k\, d\chi = \Delta k\, du$

$$\psi(x) = \underbrace{\frac{1}{\sqrt{\sqrt{2\pi}\,\Delta x}}}_{\text{norm}} \underbrace{e^{-\frac{1}{4}\left(\frac{x}{\Delta x}\right)^2}}_{\text{envelope}} \underbrace{e^{ik_0 x}}_{\text{phase}}$$

compare:

$$A(k) = \underbrace{\frac{1}{\sqrt{\sqrt{2\pi}\,\Delta k}}}_{\text{norm}} \underbrace{e^{-\frac{1}{4}\left(\frac{k-k_0}{\Delta k}\right)^2}}_{\text{envelope}}$$



Question: How does $\psi(x)$ change if $A(k) \to A(k) e^{-ikx_0}$?

Note: We use the inverse Fourier transform for $\psi(x) \to A(k)$
The Gaussian wave packet is symmetric in x vs. k-space!

* Heisenberg uncertainty principle: the more frequency components of a wave packet, ie the larger the bandwidth Δk, the easier it is to localize $\psi(x)$, ie the smaller Δx.

The Gaussian is the "minimum uncertainty wave packet" for which $\Delta x \Delta k = \frac{1}{2}$. Any other shape has larger $\Delta k \Delta x$.

Thus H.U.P. $\Delta x \Delta k \geq \frac{1}{2}$ or $\boxed{\Delta x \Delta p \geq \hbar/2}$