L08-Wave equation,impedance, and dispersion relations

Sunday, September 4, 2016 10:04

* Recall the definition of a wave from L05:
  - A wave is a collective mode of oscillation that can transfer energy and momentum from one location to another without the corresponding transfer of material (mass)

* We studied the wave function last time (kinematics) now let's look at its evolution (dynamics), which is governed by the wave equation.

  Linear differential equations allow superposition of waves, which allows for constructive & destructive interference.

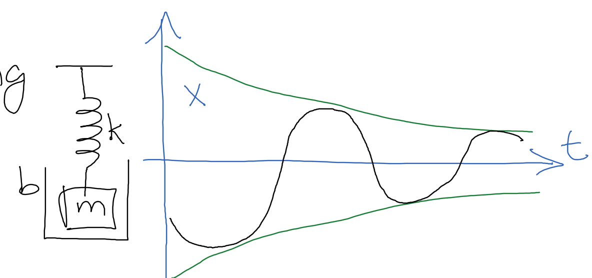
  ex: "$\frac{\partial}{\partial x}$" is a linear operator: $\partial_x [\psi_1 + \psi_2] = \frac{\partial}{\partial x}\psi_1 + \frac{\partial}{\partial x}\psi_2$

  Homogeneous differential equations are independent of position, and have pure frequency waves as solutions.

  Isotropic no preferred direction in space, applies to vector fields.

* Let's build up the wave equation from 1-particle (oscillation) to n-particles (eigenvectors representing modes of oscillation) to a continuous medium (eigenfunctions ~ waves of oscillation)

  A) 1-particle: damped mass on a spring



$$\vec{F} = -kx - b\dot{x} + F_{ext}(t) = m\ddot{x}$$

$$\underbrace{[m\partial_t^2 + b\partial_t + k]}_{\text{linear operator "L"}} x = F_{ext}(t)$$

Note that $\underbrace{\boxed{\partial_t}}_{\text{operator}} e^{-i\omega t} = \underbrace{\boxed{-i\omega}}_{\text{eigenvalue}} \underbrace{e^{-i\omega t}}_{\text{eigenfunction}}$ so we can replace $\boxed{\partial_t} \to \boxed{-i\omega}$ differential $\to$ algebraic

Homogeneous equation: (transient) $Le^{-i\omega t} = \underbrace{[-m\omega^2 - ib\omega + k]}_{\text{characteristic polynomial}} e^{-i\omega t} = 0$ $\quad b=0: \omega_{00}^2 = k/m$

2 solutions for 2nd order equation (2 constants of integration)

$$\omega = \frac{-ib}{2m} \pm \sqrt{\left(\frac{-ib}{2m}\right)^2 + \frac{k}{m}} = -i/\tau \pm \omega_0 \qquad x = x_0 e^{-t/\tau}[A\cos\omega_0 t + B\sin\omega_0 t]$$

if underdamped: $1/\tau < \omega_{00}$

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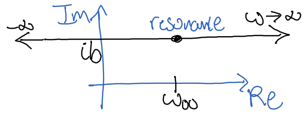
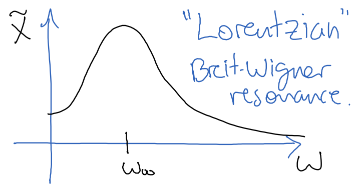
Particular solution: include the source (driving) term $F_{ext}(t)$

$F_{ext}(t) = \int d\omega\, \tilde{F}(\omega)\, e^{-i\omega t}$ let $x(t) = \int d\omega\, \tilde{x}(\omega)\, e^{-i\omega t}$

then $\int d\omega \left\{ \left[-m\omega^2 - ib\omega + k\right] \tilde{x}(\omega) = \tilde{F}(\omega) \right\} e^{-i\omega t}$

$$\tilde{x}(\omega) = \frac{\tilde{F}(\omega)}{m(\omega^2 - \omega_{00}^2) - ib\omega}$$





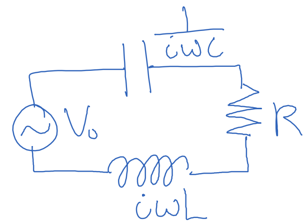
This is essentially the method of Laplace transforms $\int ds\, e^{st}$ $s = -i\omega$

The quantity $Z \equiv b + i\left(m\omega - \frac{k}{\omega}\right)$ is called "mechanical impedance" and determines the amplitude of oscillation (actually v_{max}) as a function of the applied force: $F = v \cdot Z$

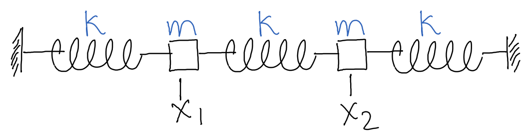
There is an electrical analog, the "tank circuit"

$V = IR \rightarrow IZ$ (complex impedance) $Z = R + i\omega L + \frac{1}{i\omega C}$

$F = vb \Rightarrow vZ$ (mech. impedance) $Z = b + i\omega m + \frac{k}{i\omega}$



B) System of n masses on springs: one "F=ma" per mass:



$k(x_2 - x_1) - kx_1 = m\ddot{x}_1$

$k(x_1 - x_2) - kx_2 = m\ddot{x}_2$

$$k\begin{pmatrix}-2 & 1 \\ 1 & -2\end{pmatrix}\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix} = -m\omega^2\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix} \qquad \begin{pmatrix}x_1 \\ x_2\end{pmatrix} = \begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix} e^{-i\omega t}$$

(operator; eigenvalue; eigenvector (mode))

let $\lambda = \omega^2/\omega_0^2 = \frac{\omega^2}{k/m}$

$(2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3$

$= (\lambda - 3)(\lambda - 1) = 0$

$$x(t) = \tilde{x}_1 \begin{pmatrix}1 \\ 1\end{pmatrix} e^{-i\omega_0 t} + \tilde{x}_2 \begin{pmatrix}1 \\ 1\end{pmatrix} e^{-i3\omega_0 t}$$

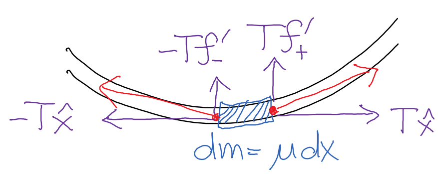
The solution is $x_i(t)$, "i" indexes the mass number.

C) Continuous distribution of masses and forces:

Now the solution is $f(x,t)$, where "x" indexes the infinitessimal mass

Wave on a string:

The balanced tension keeps the rope from moving left/right, but curvature $\left(\frac{\partial^2}{\partial x^2}\right)$ causes acceleration $\Rightarrow$ oscillation up/down



$dF = dm\, a$

$d(Tf') = \mu dx \cdot \ddot{f}$

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but curvature ($\frac{\partial^2}{\partial x^2}$) causes
acceleration → oscillation up/down

$$dF = dm\, a$$
$$d(Tf') = \mu dx \cdot \ddot{f}$$
$$Tf'' = \mu \ddot{f}$$

$$\underbrace{T\frac{\partial^2}{\partial x^2} - \mu\frac{\partial^2}{\partial t^2}}_{\text{operator}} f(x,t) = 0 \qquad \text{let } f(x,t) = A\underbrace{e^{i(kx-\omega t)}}_{\text{eigenfunction}}$$

$$[-Tk^2 + \mu\omega^2]\, A e^{i(kx-\omega t)} = 0$$

* $\mu\omega^2 = Tk^2$ is the disperion relation $\omega(k)$ (characteristic equation)

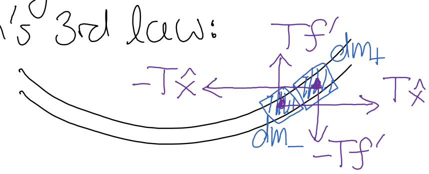
a) it determines the phase velocity $v_\phi = \omega/k$ for each k
b) it determines the group velocity $v_g = d\omega/dk$ of a pulse
c) it determines the dispersion $\beta = d^2\omega/dk^2$ (spreading out)

In general, the homogenous wave equation is equivalent to its dispersion relation, since ∂_x and ∂_t can be replaced by ik and $-i\omega$.

$$\mathcal{O}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) A_0 e^{i(kx-\omega t)} = \mathcal{O}(ik_x, -i\omega)\, A_0 e^{i(kx-\omega t)} = 0 \quad \text{[wave eq.]}$$

$$\Rightarrow \quad \mathcal{O}(ik_x, -i\omega) = 0 \quad \text{[dispersion relation]}$$

* Vertical kinetic energy is transferred along the string as momentum is exchanged according to Newton's 3rd law:



$$F_{-\text{on}+} = -F_{+\text{on}-} = \underset{\text{exchanged}}{\Delta p} = Tf'\Delta t$$

Power transferred: $P = F \cdot v = Tf'\dot{f}$ or $Zv^2 = Z\dot{f}^2$

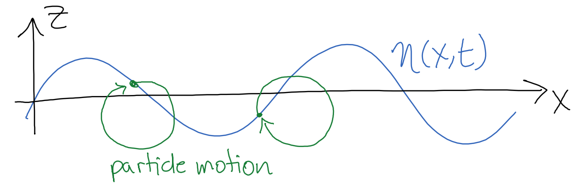
where "Mechanical Wave Impedance": $Z \equiv F/v = \frac{Tf'}{\dot{f}} = \frac{Tk}{\omega} = \sqrt{T\mu}$

analogous to "Electrical Impedance" $Z \equiv V/I$ of the telegraph eq.

Thus impedance is also important for wave propagation and afects the transfer of energy, but not the velocity.

Impedance mismatch is responsible for reflection of waves from an interface of 2 media.

* Gravity wave (water) ignoring surface tension, and infinitely deep water:



Let $\eta(x,t)$ be the height of the wave above equilibrium. and $\phi(x,z,t)$ be a flow potential of water at height z. It's gradient is the velocity of the water $\vec{v}(x,z,t) = \nabla\phi$. $\phi(x,z,t)$ can be formed if the flow is irrotational, ie. $\nabla\times\vec{v} = 0$. The water is incompressible: $\nabla\cdot\vec{v} = \nabla^2\phi = 0$

The solution to Laplace's equation $\nabla^2\phi = 0$

is $\phi = \frac{\omega A}{k} e^{-kz} \sin(kx-\omega t)$, which satisfies the

boundary conditions $\phi \to 0$ as $z \to -\infty$ (far from surface) and $v_z = \partial_z\phi = \partial_t\eta$ at $z=0$ (on the surface)

where $\eta(x,t) = A\cdot\cos(kx-\omega t)$

To get the dispersion relation, apply the Bernoulli equation:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = -\nabla PV + m\vec{g}$$

Divide by mass and integrate over height z.

$$\int dz\left[\frac{\partial}{\partial t}\nabla\phi = -\frac{\nabla P}{\rho} + \vec{g}\right] \qquad \rho \equiv \frac{m}{V}$$

$$\frac{\partial\phi}{\partial t} = \underbrace{g\eta}_{\text{gravity}} - \underbrace{\frac{\gamma}{\rho}\frac{\partial^2}{\partial x^2}\eta}_{\text{surface tension}}$$

Ignore surface tension and substitute ϕ, η at the surface $z=0$

$$\left(\frac{\omega^2}{k} - g\right) A\cos(kx-\omega t) = 0$$

Dispersion relation: $\omega = \sqrt{gk}$

$$v_\phi = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} \qquad v_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{v_\phi}{2}$$

Different wavelengths have different velocities!