L10-Probability Currents: Unitarity

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* Probabilistic interpretation of wave function:
  - $\Psi(x)$ = probability amplitude (quantum interference)
  - $|\Psi(x)|^2$ = probability density (classical behavior)
  - $\Psi^*(x)\Psi(x)\,dx = |\Psi(x)|^2 dx$ probability of being observed
  - Imediately after observation of position, the wave function collapses to sharply peaked around x_0: $\Psi(x) = \delta(x-x_0)$ so that a second measurement will observe the same result.

* When a particle is left alone, $\Psi(x)$ evolves according to the T.D.S.E. shifting probabilidy density: http://www.brainflux.org/java/classes/Schrodinger1D.html

* Conservation of probability: (Griffiths 1.4, Gasiorowicz 2-6)

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx |\Psi|^2 = \int_{-\infty}^{\infty} dx \frac{\partial}{\partial t}|\Psi(x,t)|^2 \quad \text{[move } \tfrac{d}{dt} \text{ inside } \textstyle\int dx]$$

$$\partial_t\left(|\Psi|^2 = \Psi^*\Psi\right) = \Psi^* \cdot \partial_t \Psi + \partial_t \Psi^* \cdot \Psi \quad \text{[product rule]}$$

Now use the TDSE to convert $\frac{\partial}{\partial t}$ to $\frac{\partial^2}{\partial x^2}$, which we can integrate:

$$\Psi^*\left[\partial_t \Psi = \frac{i\hbar}{2m}\partial_x^2 \Psi - \frac{i}{\hbar} V \Psi\right]$$

[schrödinger]

$$\Psi\left[\partial_t \Psi^* = \frac{-i\hbar}{2m}\partial_x^2 \Psi^* + \frac{i}{\hbar} V^* \Psi^*\right]$$

$*$: $i \to -i$
$\Psi = \Psi_{real} + i\,\Psi_{im}$
$\Psi^* = \Psi_{real} - i\,\Psi_{im}$
$V^* = V$ (real)

Substitute both of these into the above equation:

$$\underbrace{\partial_t |\Psi|^2}_{\rho(x)} = \frac{i\hbar}{2m}\left(\Psi^* \partial_x^2 \Psi - \partial_x^2 \Psi^* \cdot \Psi\right)$$

$$= \frac{\partial}{\partial x}\underbrace{\left(\Psi^* \cdot \frac{i\hbar}{2m}\partial_x \Psi \overbrace{- \frac{i\hbar}{2m}\partial_x \Psi^* \cdot \Psi}^{\text{"+ c.c." (complex conjugate)}}\right)}_{j(x)}$$

- The $\frac{\partial}{\partial x}$ in the second line has 2 extra terms, $\partial_x \Psi^* \partial_x \Psi$ which cancel.

- The two terms ensure that $j(x)$ is real: $\frac{1}{2}(z+z^*) = \frac{1}{2}[(x+iy)+(x-iy)] = x = Re[z]$ just like $\rho(x) = \Psi^*(x)\Psi(x)$ is real: $zz^* = (x+iy)(x-iy) = x^2+y^2 = r^2 = |z|^2$

- Note that $\partial_t |\Psi|^2$ is a perfect differential of "$j(x)$". Thus:

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx |\Psi|^2 = \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} j(x) = j(x)\Big|_{-\infty}^{\infty} \to 0 \qquad \Psi(\pm\infty) \to 0$$

- The invariance of probability under evolution of the T.D.S.E. is conceptually similar to the length of a vector being invariant under rotations. We say the "evolution operator" is "unitary".

* Continuity equation: recall $\rho(x) = |\Psi(x)|^2 = \Psi^*\Psi$ [probability density]

Define the "momentum operator" $\hat{p} \equiv -i\hbar\partial_x$ i.e. $\hat{p}e^{ikx} = \underbrace{\hbar k}_{\text{eigenvalue } p=\hbar k} e^{ikx}$

Then $$\partial_t (\underbrace{\Psi^*\Psi}_{\rho(x)}) + \partial_x \Big(\underbrace{\Psi^* \frac{\hat{p}}{2m}\Psi - \Psi \frac{\hat{p}}{2m}\Psi^*}_{\vec{j}(x)}\Big) = 0$$ no time derivatives!

This equation is the 1-dimensional [we could have derived it in 3-d] form of a "continuity equation" $\partial_t \rho + \nabla\cdot\vec{j} = 0$, which is the mathematical way of saying that something is conserved.

$\rho(\vec{r})$ = the probability density = $\frac{\text{probability}}{\text{volume}}$ $\quad \int_V \rho(\vec{r})d\tau$ = probability

$\vec{j}(\vec{r})$ = probability current [density] = $\frac{\text{probability}}{\text{area}\cdot\text{time}}$ $\quad \oint_{\partial V} \vec{j}(\vec{r})\cdot d\vec{a} = \frac{\text{probability}}{\text{time}}$

"The decrease in probability" = "the probability that leaves"

$$\frac{d}{dt}\int_V \rho(\vec{r})d\tau = \int_V \underline{\partial_t \rho(\vec{r})}\, d\tau \quad = \quad \oint_{\partial V} \vec{j}(\vec{r})\cdot d\vec{a} = \int_V \underline{\nabla\cdot\vec{j}(\vec{r})}\, d\tau$$

Thus $\partial_t \rho(\vec{r}) = \nabla\cdot\vec{j}(\vec{r})$ is the differential version of conservation of probability.

* This is similar to the Poynting theorem in E&M: dynamical equations

$$\partial_t u = \partial_t\left(\tfrac{1}{2}\vec{D}\cdot\vec{E} + \tfrac{1}{2}\vec{B}\cdot\vec{H}\right) = \frac{\partial\vec{D}}{\partial t}\cdot\vec{E} + \frac{\partial\vec{B}}{\partial t}\cdot\vec{H} = (\nabla\times\vec{H} - \vec{J})\cdot\vec{E} + (-\nabla\times\vec{E})\cdot\vec{H}$$

$$= -J\cdot E - \nabla\cdot(\vec{E}\times\vec{H}) = -\frac{dw}{dt} - \nabla\cdot\vec{S} \quad \leftarrow \text{conserved current.}$$

Just as $\vec{S} = \vec{E}\times\vec{H} = ZH^2\hat{k}$ [E&M]
or $P = Fv = Zv^2$ [string]
} the energy flux is the product of two quantities present in boundary conditions,

in Q.M., $\vec{j}(\vec{r}) = \Psi^* \cdot \frac{\hat{p}}{2m}\Psi - \Psi\frac{\hat{p}}{2m}\Psi^*$ } the boundary conditions are: $\Delta\Psi = 0$, $\Delta\partial_x\Psi = 0$

Other conserved quantities are: charge $\partial_t\rho + \nabla\cdot\vec{j} = 0$ and momentum $\partial_t\vec{p} + \nabla\cdot\overleftrightarrow{T} = 0$

* Ex: the probability density and current for a plane wave are:

$$\Psi = Ae^{\underbrace{i(kx-\omega t)}_{\phi}} \qquad \Psi^* = A^* e^{\underbrace{-i(kx-\omega t)}_{\phi}} \qquad \partial_x\Psi = ik\Psi$$

$$\rho(x) = |\Psi|^2 = \Psi^*\Psi = A^*A\, e^{i\cancel{(\phi-\phi)}} = |A|^2 = \text{const!}$$

$$\vec{j}(x) = \Psi^*\frac{-i\hbar}{2m}\partial_x\Psi - \Psi\frac{-i\hbar}{2m}\partial_x\Psi^*$$

$$= \Psi^*\frac{\hbar k}{2m}\Psi - \Psi\frac{-\hbar k}{2m}\Psi^*$$

$$= \frac{\hbar k}{m}|A|^2 = \vec{v}|A| = \rho(x)\,\vec{v}$$

[This is also a classical result:]

$$\underbrace{\frac{\text{probability}}{\text{area}\cdot\text{time}}}_{\vec{j}} = \underbrace{\frac{\text{probability}}{\text{volume}}}_{\rho} \cdot \underbrace{\frac{\text{distance}}{\text{time}}}_{\vec{v}}$$