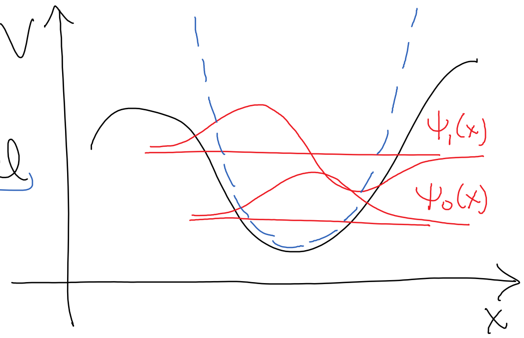
* one of the most important potentials: periodic motion is usually $\underbrace{\text{vibrational}}_{\text{approximated by harmonic potential}}$, or $\underbrace{\text{rotational}}_{\text{rigid body rotor - ch4}}$



$$V(x) = \underbrace{V(x_0)}_{\text{irrelevant}} + \underbrace{V'(x_0)(x-x_0)}_{\text{see HO5}}{}^{0} + \frac{1}{2}V''(x_0)\underbrace{(x-x_0)^2}_{\text{SHO}} \rightarrow \frac{1}{2}m\omega^2 x^2$$

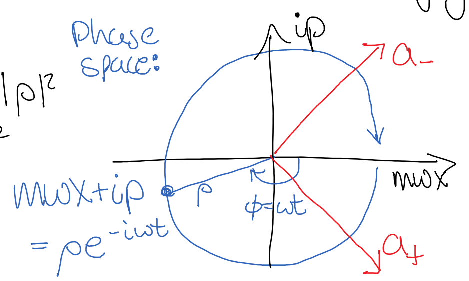
* classical solution: $F_{ext} = m\ddot{x} + b\dot{x} + kx$ let $x = x_0 e^{i\omega t}$
  if $F_{ext} = 0$ then $-m\omega^2 + ib\omega + k = 0$ $\omega = \frac{ib}{2m} \pm \sqrt{\left(\frac{ib}{2m}\right)^2 + \frac{k}{m}}$
  if $b=0$ (undamped) then $k = m\omega^2$, where ω = classical frequency

* algebraic method: (also ang. momentum & SUSY potentials!)
  factorize the potential into a product of canonical conjugates

$$H = \frac{1}{2m}\left(p^2 + (m\omega x)^2\right) = (\overbrace{iu+v}^{\sim a_-})(\overbrace{-iu+v}^{\sim a_+}) = |\rho|^2$$

classical: $T = u^2 = \frac{p^2}{2m}$ $V = v^2 = \frac{1}{2}m\omega^2 x^2$

let $a_{\pm} = \frac{1}{\sqrt{\hbar\omega}}(\mp iu + v) = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$



* note: Dirac did similar to $E^2 + p^2 = m^2$ to get the Dirac equation.

but $H \neq a_- a_+$ quantum mechanically because x, p dont commute

* canonical commutation relationship: $\boxed{[x,p] = i\hbar}$

$$\underbrace{[x,p]}\psi(x) \equiv (xp - px)\psi(x) = [x(-i\hbar\partial_x) - (-i\hbar\partial_x)x]\psi(x)$$
$$= i\hbar[-x\partial_x\psi + \partial_x(x\psi)] = i\hbar(-x\psi' + \psi + x\psi') = \underbrace{i\hbar}\psi(x)$$

* Properties of the commutator $\boxed{[A,B] = AB - BA}$: "antisymmetric derivation"

a) antisymmetric (like cross product, opposite inner product)

$$[A,B] = AB - BA = -(BA - AB)$$ (similar to the cross product)

$$[A,B] = AB - BA = -(BA - AB) = -[B,A]$$

(similar to the cross product)

compare: $a \times b = -b \times a$

b) bilinear (like the cross & inner products, and derivatives)

$$[\alpha A + \beta B, C] = (\alpha A + \beta B)C - C(\alpha A + \beta B)$$
$$= \alpha(AC - CA) + \beta(BC - CA)$$
$$= \alpha [A,C] + \beta [B,C]$$

compare: $\partial_c(\alpha a + \beta b) = \alpha \partial_c a + \beta \partial_c b$

also:

$$[A, \beta B + \gamma C] = \beta [A,B] + \gamma [A,C]$$

compare: $\partial_a(\beta b + \gamma c) = \beta \partial_a b + \gamma \partial_a b$

c) product rule – acts similar to a derivative! ["derivation"]

$$[a, bc] = (abc - bca) = (abc \underbrace{- bac + bac}_{\text{insert these for symmetry.}} - bca)$$

$$\boxed{[a, bc] = [a,b]c + b[a,c]}$$

compare: $\partial_a(bc) = (\partial_a b)c + b(\partial_a c)$

d) Jacobi identity.

compare: $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$$

* use this to calculate commutators of $a_-, a_+, \mathcal{H}$

$$a_+ a_- = \frac{1}{2m\hbar\omega}(-ip + m\omega x)(ip + m\omega x) = \frac{1}{\hbar\omega}\underbrace{\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)}_{\mathcal{H}} + \frac{i}{2\hbar}\underbrace{[x,p]}_{i\hbar}$$

$$= \mathcal{H}/\hbar\omega - 1/2 \quad \text{or} \quad \boxed{\mathcal{H} = \hbar\omega\left(a_+ a_- + \tfrac{1}{2}\right)}$$

$a_+ a_- \equiv N$

$$a_- a_+ = \mathcal{H}/\hbar\omega + 1/2 \quad \text{or} \quad \boxed{\mathcal{H} = \hbar\omega\left(a_- a_+ - \tfrac{1}{2}\right)}$$

$a_- a_+ = N + 1$

thus $\boxed{[a_-, a_+] = 1}$ $\boxed{[\mathcal{H}, a_\pm] = \pm\hbar\omega a_\pm}$ "ladders"

$$[\mathcal{H}, a_\pm] = \hbar\omega[a_+ a_-, a_\pm] = \hbar\omega\left(\underbrace{[a_+, a_\pm]}_{0 \text{ or } -1} a_- + a_+ \underbrace{[a_-, a_\pm]}_{1 \text{ or } 0}\right) = \pm\hbar\omega\, a_\pm$$

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