L17-SHO: Hermitian adjoint

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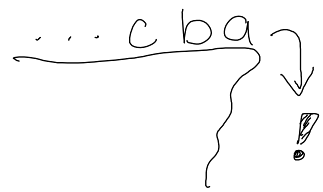
* Side concept #1 "annihilation operator"



- Conversation between Spencer & Madison (6yrs old)

M: Spencer, why did you do that?
S: Curtis told me to.
M: Would you jump off a cliff if Curtis told you to?
S: No, I don't know where any cliffs are.
M: I think cliffs are onl in the African Savannah.
S: Yeah, I think so too.

- The following "lemming matrix" marches components off the "edge of a cliff" (the top of the vector)

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b \\ c \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Action on unit vectors: $\hat{z} \xrightarrow{A} \hat{y} \xrightarrow{A} \hat{x} \xrightarrow{A} 0$

- Example: $\frac{d}{dx}(a + bx + cx^2 + \dots) = (b + 2cx + \dots)$

* Side Concept #2: Hermitian adjoint (transpose & conjugate)

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow A^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad 0 \xleftarrow{A^\dagger} \hat{z} \xleftarrow{A^\dagger} \hat{y} \xleftarrow{A^\dagger} \hat{x}$$

180°

- The transpose matrix "multiplies to the left"

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right]^T = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

(the resulting matrices are equal)

- The Hermitian adjoint changes a matrix A by taking its transpose and complex conjugate. A^{T*}

- As above, the transpose A^T applies to each factor, but reverses the order of multiplication. (similar to the inverse of a square matrix A^{-1}

- Complex conjugation also applies to each factor, A^* but keeps the order the same

- The combined result: $(ABC)^\dagger = (C^T B^T A^T)^* = C^{T*} B^{T*} A^{T*} = C^\dagger B^\dagger A^\dagger$

$$\boxed{(ABC)^\dagger = C^\dagger B^\dagger A^\dagger} \quad \text{and} \quad (ABC)^{-1} = C^{-1} B^{-1} A^{-1}$$

- The Hermitian adjoint of a complex number (1x1 matrix) (including the result of an inner product) is just its complex conjugate

$$\boxed{c^\dagger = c^*} \qquad (\langle\psi|\varphi\rangle)^\dagger = (\langle\psi|\varphi\rangle)^* \qquad \left[(v_x^* \; v_y^*)\begin{pmatrix} w_x \\ w_y \end{pmatrix}\right]^\dagger = (w_x^* \; w_y^*)\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

Transposing the scalar product $(x\,x^*)\begin{pmatrix} x \\ x^* \end{pmatrix}$ didn't change the result, only complex conjugation did.

- Applying the Hermitian adjoint to abstract vectors:

$$\Big[\langle\psi|\varphi\rangle \equiv \underbrace{\int dx}_{\text{sum over}}\, \psi^*\underbrace{(x)}_{\text{index}}\,\varphi\underbrace{(x)}\Big]^\dagger \iff \vec{v}\cdot\vec{w} = (v_x^* \; v_y^*)\begin{pmatrix} w_x \\ w_y \end{pmatrix} = \underbrace{\sum_i}_{\text{sum}} w_{\underbrace{i}_{\text{index}}}^* v_{\underbrace{i}}$$

$$\| \qquad\qquad \|$$

$$\langle\varphi|\psi\rangle = \int dx\, \varphi^*(x)\psi(x)$$

Think of "$\cdot$" as "$T*$" acting on the left.

Thus $\langle\psi|\varphi\rangle^\dagger = |\varphi\rangle^\dagger \langle\psi|^\dagger = \langle\varphi|\psi\rangle$ (the complex conjugate)

Breaking it up: $\boxed{|\psi\rangle^\dagger = \langle\psi|}$ and $\boxed{\langle\psi|^\dagger = |\psi\rangle}$

just like column vectors: $\begin{pmatrix} v_x \\ v_y \end{pmatrix}^\dagger = (v_x^* \; v_y^*)$ and row vectors: $(v_x v_y)^\dagger = \begin{pmatrix} v_x^* \\ v_y^* \end{pmatrix}$

just like column vectors: $\begin{pmatrix} V_x \\ V_y \end{pmatrix}^\dagger = (V_x^* \; V_y^*)$ and row vectors: $(V_x V_y)^\dagger = \begin{pmatrix} V_x^* \\ V_y^* \end{pmatrix}$

- To calculate the adjoint of an abstract operator, consider how it acts between a dot product, or on a vector which gets transposed:

$$\langle \psi | A^\dagger | \varphi \rangle = \langle \varphi | A | \psi \rangle^* \quad \text{or} \quad \langle \psi | A^\dagger \varphi \rangle \equiv \langle A \psi | \varphi \rangle$$

$$\text{or } A^\dagger | \varphi \rangle \equiv \left(|\varphi\rangle^\dagger A \right)^\dagger = \left(\langle \varphi | A \right)^\dagger \quad \text{or } \langle \psi | A^\dagger \equiv \left(A | \psi \rangle \right)^\dagger$$

- Example: $\left(\frac{d}{dx}\right)^\dagger$ using $\langle \psi | D | \varphi \rangle \equiv \int dx\, \psi^*(x) \frac{d}{dx} \varphi(x)$

$$= \int \psi^* d\varphi = \cancel{\psi^* \varphi \Big|_{-\infty}^{\infty}}^{\,0} - \int \varphi \, d\psi = \int dx\, \varphi(x) \frac{-d}{dx} \psi^*(x)$$

$$= \left(\langle \varphi | D^\dagger | \psi \rangle \right)^* . \quad \text{Comparing terms, } \boxed{\left(\frac{d}{dx}\right)^\dagger = \frac{-d}{dx}}$$

Thus both position and momentum operators are Hermitian!

$$\boxed{x^\dagger = x} \qquad p^\dagger = -i\hbar \frac{\partial}{\partial x} = +i\hbar \frac{-\partial}{\partial x} = p \qquad \boxed{p^\dagger = p}$$

- Def'n "Hermitian" operator: $\boxed{A^\dagger = A}$

This type of operator has real eigenvalues and represents physical measurements.

* Overview of the SHO, two methods:

A) operator algebra: $\mathcal{H} = \hbar\omega(a_+ a_- + \frac{1}{2})$ $\quad [a_-, a_+] = 1$ $\quad [\mathcal{H}, a_\pm] = \pm\hbar\omega a_\pm$

1) $\mathcal{H}|n\rangle = E_n |n\rangle$ $\quad \mathcal{H} a_\pm |n\rangle = a_\pm \mathcal{H} |n\rangle \pm \hbar\omega \, a_\pm |n\rangle = (E_n \pm \hbar\omega) a_\pm |n\rangle$

thus $a_\pm |n\rangle \propto |n \pm 1\rangle$ and $E_n = \hbar\omega(n + \frac{1}{2})$, ie $a_+ a_- = n$

2) $a_-^\dagger = a_+ \Rightarrow$ $n = \langle n | a_+ a_- | n \rangle = \langle a_- n | a_- n \rangle \Rightarrow$ $a_- |n\rangle = \sqrt{n}\,|n-1\rangle$

$n+1 = \langle n | a_- a_+ | n \rangle = \langle a_+ n | a_+ n \rangle \Rightarrow$ $a_+ |n\rangle = \sqrt{n+1}\,|n+1\rangle$

we can now write matrices for a_-, a_+ $\hat{x}, \hat{x}^2, \hat{V}$ $\hat{p}, \hat{p}^2, \hat{T}$ $\hat{H}$ [H06]

A)→B) transition: dimmensionless co-ordinates

$$a_\pm = \frac{1}{\sqrt{2\hbar\omega m}}(\mp i\hat{p} + m\omega x) = \frac{1}{\sqrt{2}}(\mp \partial_\xi + \xi) \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

3) $a_-|0\rangle = 0$ $\quad (\partial_\xi + \xi)\psi_0 = 0 \quad \psi_0 = \pi^{-1/4} e^{-\frac{1}{2}\xi^2}$ Gaussian!

4) $$|n\rangle = \frac{1}{\sqrt{n!}} a_+^n |0\rangle = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}}(-\partial_\xi + \xi)^n e^{-\frac{1}{2}\xi^2} = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} \underbrace{H_n(\xi)}_{\text{Hermite polys}} e^{-\frac{1}{2}\xi^2}$$

$H_0 = 1 \quad H_1 = 2\xi \quad H_2 = 4\xi^2 - 2 \quad H_3 = 8\xi^3 - 12\xi$

B) Frobenius method: Taylor series : ODE + B.C.'s.

5) dimensionless $\mathcal{H}$: $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right)\psi = E\psi \Rightarrow \left(-\frac{d^2}{d\xi^2} + \xi^2 - K\right)\psi = 0$

6) asymptotic form: $\psi = h(\xi)e^{-\xi^2/2}$ $\qquad h'' - 2\xi h' + (K-1)h = 0$

7) power series: $h(\xi) = \sum_{j=0}^{\infty} a_j \xi^j \qquad a_{j+2} = \frac{2j+1-K}{(j+1)(j+2)}$

8) B.C.'s (truncation): $E_n = \frac{1}{2}\hbar\omega K_n = \hbar\omega(n+\frac{1}{2}) \qquad a_{j+2}^{(n)} = \frac{-2(n-j)}{(j+1)(j+2)}$ [H07]
quantization,
standardization $\qquad \psi_n(\xi) = \left(\frac{m\omega}{\pi\hbar}\right) \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}$ [Hermite]