L18-SHO: Operator algebra

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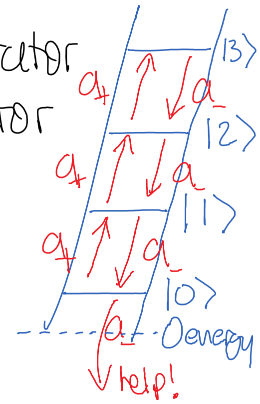
* eigenvalues: let $\mathcal{H}|n\rangle = E_n|n\rangle$, use $[\mathcal{H}, a_\pm] = \pm\hbar\omega\, a_\pm$

then $\mathcal{H}\, a_\pm|n\rangle = (a_\pm \mathcal{H} \pm \hbar\omega\, a_\pm)|n\rangle = (E_n \pm \hbar\omega)\, a_\pm|n\rangle$
and $a_\pm|n\rangle$ is a different eigenstate with higher/lower energy

thus $a_\pm$ are called ladder operators
$a = a_-$ is called lowering or annihilation operator
$a^\dagger = a_+$ is called raising or creation operator



let $|0\rangle$ be the ground state (lowest energy)
then $|n\rangle = A_n\, a_+^n|0\rangle$ $\quad \underbrace{a_-|0\rangle}_{\text{unit vector}} = 0$ ← origin: 0 vector

$$\mathcal{H}|0\rangle = \hbar\omega(\underbrace{a_+a_-}_{n} + \tfrac{1}{2})|0\rangle = \tfrac{1}{2}|0\rangle \quad \text{thus} \quad E_n = \hbar\omega(n + \tfrac{1}{2})$$

* normalization: let $a_+|n\rangle = c_n|n+1\rangle$ $\quad a_-|n\rangle = d_n|n-1\rangle$

note: $a_+a_-|n\rangle = \mathcal{H}/\hbar\omega - \tfrac{1}{2}\,|n\rangle = n|n\rangle$ $\quad a_-a_+|n\rangle = n+1\,|n\rangle$

$$n = \langle n|n|n\rangle = \langle n|\overleftarrow{a_+}\,\overrightarrow{a_-}|n\rangle = \langle n+1|c_n^* c_n|n+1\rangle = |c_n|^2$$

$$n = \langle n-1|n+1|n-1\rangle = \langle n-1|a_-a_+|n-1\rangle = \langle n-1|d_n^* d_n|n-1\rangle = |d_n|^2$$

thus
$$\boxed{\begin{aligned} a_-|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a_+|n-1\rangle &= \sqrt{n}\,|n\rangle \\ |n\rangle &= \tfrac{1}{\sqrt{n!}}\, a_+^n|0\rangle \end{aligned}}
\qquad a_+ \sim \begin{pmatrix} 0 & 0 & & & \\ \sqrt{1} & 0 & 0 & & \\ & \sqrt{2} & 0 & 0 & \\ & & \sqrt{3} & 0 & \\ & & & \ddots & \end{pmatrix}
\qquad a_- \sim \begin{pmatrix} 0 & \sqrt{1} & & \\ 0 & 0 & \sqrt{2} & \\ & 0 & 0 & \sqrt{3} \\ & & 0 & 0 & \ddots \end{pmatrix}$$

* in-class: calculate matrices of $x, p, [x,p], p^2, x^2, \mathcal{H}$

* Wave functions: $a_-|0\rangle = (i\hat{p} + m\omega x)\,\psi_0(x) = 0$

$$\left(-i^2\hbar \frac{d}{dx} + m\omega x\right)\psi_0(x) = 0 \qquad d\ln\psi_0 = \frac{d\psi_0}{\psi_0} = -\frac{m\omega x}{\hbar}dx = d\,\frac{-m\omega}{2\hbar}x^2$$

$$\psi_0(x) = A_0 e^{-\frac{m\omega}{2\hbar}x^2} \qquad 1 = \int_{-\infty}^{\infty} dx\, |A_0|^2 e^{-\frac{m\omega}{\hbar}x^2} = \sqrt{\frac{\hbar\pi}{m\omega}}$$

$$= \left(\frac{m\omega}{\hbar\pi}\right) e^{-\frac{m\omega}{2\hbar}x^2} \qquad \text{recall L07: } \int_{-\infty}^{\infty} e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

* Summary: key relations:

$$a_\pm = \frac{1}{\sqrt{2\hbar\omega m}}(\mp i\hat{p} + m\omega\hat{x}) = \frac{1}{\sqrt{2}}(\mp\partial_\xi + \xi) \qquad a_-^\dagger = a_+ \qquad [a_-, a_+] = [\partial_\xi, \xi] = 1$$

$$\mathcal{H} = \hbar\omega(\underbrace{a_+a_-}_{n} + \tfrac{1}{2}) \qquad \mathcal{H}|n\rangle = E_n|n\rangle \qquad [a_+a_-, a_\pm] = \pm a_\pm \qquad a_+a_- = n$$

$$n = \langle n|a_+a_-|n\rangle = \langle a_-n|a_-n\rangle \qquad \boxed{\begin{aligned} a_-|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a_+|n-1\rangle &= \sqrt{n}\,|n\rangle \end{aligned}} \qquad a_-|0\rangle = 0$$

$$n+1 = \langle n|a_-a_+|n\rangle = \langle a_+n|a_+n\rangle \qquad \langle\xi|0\rangle = e^{-\frac{1}{2}\xi^2}$$