L20-Dispersion: Propagation of a free particle

Friday, September 16, 2016 08:00

* We intuited the Schrödinger equation from the dispersion relation of the free particle
  Now we will work backwards and solve the free particle.
  We will use the usual techniques to analyze the time dependent evolution of the wave function/packet.

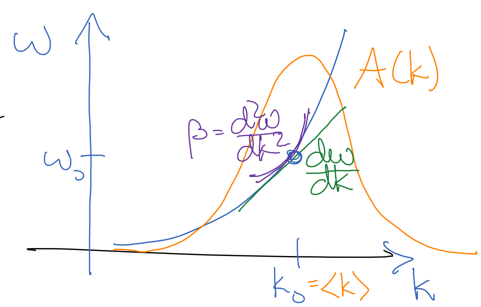
* $V=0$: $-\frac{\hbar^2}{2m}\psi''(x) + 0 = E\psi = \frac{\hbar^2k^2}{2m}\psi \qquad \psi_k(x) = e^{ikx}$

  $\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\, A(k)\, e^{i(kx-\omega(k)t)}$ where $A(k)$ comes from $\Psi(x,0)$

* How does a wave packet evolve? Add the time dependence:

  $e^{ikx} \to e^{i(kx-\omega t)}$ of each frequency component, and

  Taylor expand the dispersion relation



$$\omega(k) \approx \underbrace{\omega_0}_{\omega(k_0)} + \frac{d\omega}{dk}\Big|_{k_0}(k-k_0) + \frac{1}{2}\frac{d^2\omega}{dk^2}\Big|_{k_0}(k-k_0)^2$$

$$kx - \omega t = k_0x + (k-k_0)x - \omega(k)t$$

$$\approx k_0\Big(x - \underbrace{\frac{\omega_0}{k_0}}_{v_\phi}t\Big) + (k-k_0)\Big(x - \underbrace{\frac{d\omega}{dk}\Big|_{k_0}}_{v_g}t\Big) - \frac{1}{2}(k-k_0)^2\underbrace{\frac{d^2\omega}{dk^2}\Big|_{k_0}}_{\beta}t$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\,\boxed{A(k)}\,e^{i(kx-\omega(k)t)}$$ * (adding the time dependence)

$$\approx \underbrace{e^{i\overbrace{k_0(x-v_\phi t)}^{\phi}}}_{\text{phase}}\cdot\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dq\,A(k_0+q)\,e^{iq\underbrace{(x-v_gt)}_{\text{translation}} - \underbrace{\frac{i}{2}q^2\beta t}_{\text{broadening}}}}_{\text{envelope (wave packet)}}$$ where $q = (k-k_0)$

The ripples travel at the phase velocity.
The "particle" is represented by the envelope, a function of x_t, which thus travels at the group velocity.
The dispersion dephases the Fourier components on the fringe of the wave packet, which causes it to spread out.

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$$\psi(x) = \underbrace{\frac{1}{\sqrt{\sqrt{2\pi}\,\Delta x}}}_{N_x = \text{norm}}\ \underbrace{e^{-\frac{1}{4}\left(\frac{x-x_0}{\Delta x}\right)^2}}_{\text{envelope}}\ \underbrace{e^{ik_0 x}}_{\text{phase}} \quad \Longleftrightarrow \quad \underbrace{\frac{1}{\sqrt{\sqrt{2\pi}\,\Delta k}}}_{N_k = \text{norm}}\ \underbrace{e^{-\frac{1}{4}\left(\boxed{\frac{k-k_0}{\Delta k}}\right)^2}}_{\text{envelope}}\ \underbrace{e^{-i\boxed{x_0}k}}_{\text{phase}} = A(k) \quad \#$$

DIFFERENCE!

$$= \frac{1}{\sqrt{2\pi}}\int dk\, A(k)\, e^{ikx} \qquad x \leftrightarrow k \qquad \frac{1}{\sqrt{2\pi}}\int dx\, \psi(x)\, e^{-ikx} =$$

DIFFERENCE!

Center the momentum: $\boxed{A(k_0+q)} = N_k e^{\underbrace{-\frac{1}{4}\frac{q^2}{\Delta k^2}}_{\text{gaussian}}\underbrace{-iqx_0}_{\text{shift}}\underbrace{-ik_0x_0}_{\text{global phase}}}$ into *

$$\Psi(x,t) = e^{i(\phi - k_0 x_0)}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dq\, N_k\, e^{-q^2\boxed{\underbrace{\left(\underbrace{\frac{1}{4\Delta k^2}}_{\Delta x^2(0)}+\frac{i}{2}\beta t\right)}_{\Delta x^2(t)}}}\, e^{iq\boxed{(x-\underbrace{(x_0+v_g t)}_{x_0(t)})}}$$ apply the above F.T. pair #

$$= e^{i\underbrace{k_0(x-v_\phi t)}_{\phi}}\, e^{-ik_0x_0}\, \frac{1}{\sqrt{\sqrt{2\pi}\,\Delta x(t)}}\, e^{\underbrace{-\frac{1}{4}\frac{(x-x_0(t))^2}{\boxed{\Delta x^2(t)}}}_{g}}$$

where $x_0(t) = x_0 + v_g t$, $\Delta x^2(t) = \Delta x^2(0) + \frac{i}{2}\beta t$

To understand the dispersion (spreading out of the wave packet), calculate the probability density $P(x) = |\psi(x)|^2$

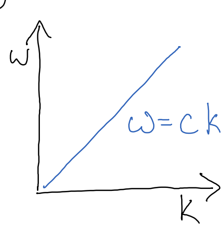
$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\,\Delta x(t)}\left|e^{-\frac{1}{4}\frac{(x-x_0)^2}{\Delta x^2(t)}}\right|^2 \qquad \left|e^{1/z}\right|^2 = e^{1/z}\cdot e^{1/z^*} = e^{\frac{z^*+z}{|z|^2}} = e^{2\,\mathrm{Re}[z]/|z|^2}$$

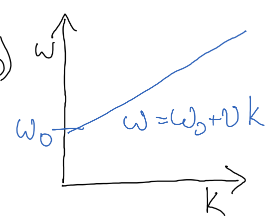
$$= \frac{1}{\sqrt{2\pi}\,\Delta x(t)}\, e^{-\frac{1}{2}(x-x_0)^2\,\mathrm{Re}[\Delta x^2(t)]/|\Delta x^2(t)|^2} = \frac{1}{\sqrt{2\pi}\,\Delta x(t)}\, e^{-\frac{1}{2}\frac{(x-x_0)^2}{\Delta x^2(0) + \beta^2 t^2/\Delta x^2(0)}}$$

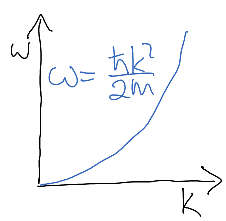
is a Gaussian distribution of width $\Delta x(t) = \Delta x(0)\sqrt{1+\frac{1}{4}\beta^2 t^2/\Delta x^4(0)}$, which spreads out in time! (dispersion)

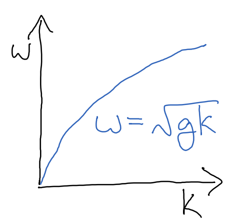
Thus the dispersion constant $\beta = \frac{d^2\omega}{dk^2}$, ie the second derivative of the dispersion relation, is actually what causes dispersion. The bending of light at different angles is due to frequency-dependent impedance $Z(k) = \sqrt{\mu/\varepsilon} = \mu v = \mu\omega/k$, which is related to dispersion.

* Types of dispersion relations:

a) 

b) 

c) 

d) 

a) nondispersive medium [E&M vacuum] constant $v_\phi = v_g = c$

b) constant v_g with extra global phase factor $e^{i\omega_0 t}$

c) free particle: $2v_\phi = v_g = \frac{\hbar k}{m} = v_p$ ripples lag behind.

d) gravity waves: $\frac{1}{2}v_\phi = v_g = \frac{1}{2}\sqrt{\frac{g}{k}}$ ripples advance