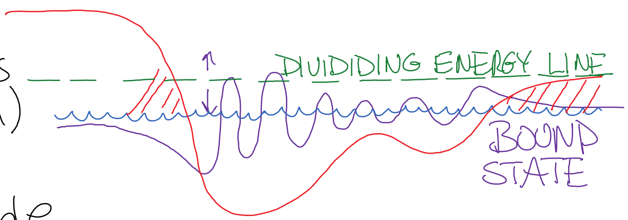
* General features of wave functions:

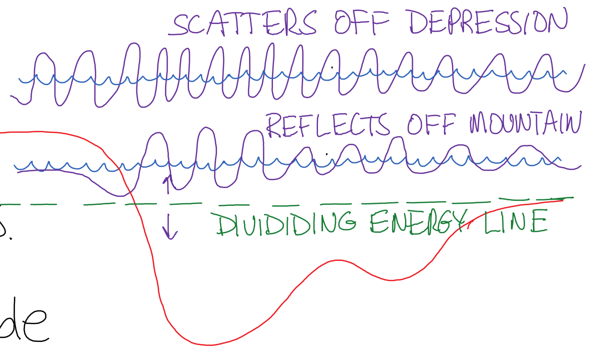
+ turning point: $V(x_{TP}) = E$ so $T(x_{TP}) = 0$
  - classical particle turns around
  - quantum wave: curvature $= k^2\psi$ switches from oscillatory: $k^2 > 0$ to exponential $k^2 = -\kappa^2 = (i\kappa)^2 < 0$

$\psi(x) = e^{\pm ikx} = \cos(kx) \pm i\sin(kx)$ to $\psi(x) = e^{\pm i(i\kappa)x} = e^{\pm\kappa x} = \cosh\kappa x \pm \sinh\kappa x$

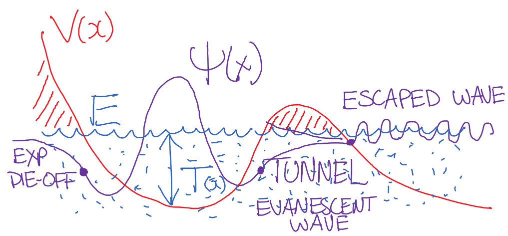
+ bound state: exponentially decays at both ends.
  - E quantized by external B.C.'s (Sturm-Liouville system)
  - discrete energy spectrum
  - eigenfunctions normalizable



+ scattering state can escape to one or both ends
  - not quantized by external B.C.'s
  - still has eigenfunctions/values.
  - continuous energy spectrum
  - eigenfunctions not normalizeable



+ tunnelling: (evanescent wave) exponential dropoff through a mountain. oscillatory with reduced amplitude on other side.



* boundary conditions narrow down the general solution to the specific one. There are two types:

a) external conditions: $\psi(x) \to 0$ fast enough as $x \to \pm\infty$

that $\int_{-\infty}^{\infty} dx\,|\psi(x)|^2 = 1$ ie. bound states must be normalizable

It is impossible to normalize scattering states; instead we require that
$\langle k | k' \rangle \equiv \int_{-\infty}^{\infty} dx\, \psi_k^*(x)\, \psi_{k'}(x) = \delta(k-k')$ for convenience
Wave packets can still normalized $\psi(x) = \int dk\, A(k)\, \psi_k(x)$

b) internal conditions: if there is a discontinuity in $V(x)$ at $x=a$, then we must obtain separate solutions $\psi(x) = \begin{cases} \psi_1(x) & x<a \\ \psi_2(x) & x>a \end{cases}$ to the TISE.
Continuity B.C.'s are used to sew these into a single wave function.
If the potential is finite, there are two continuity conditions:

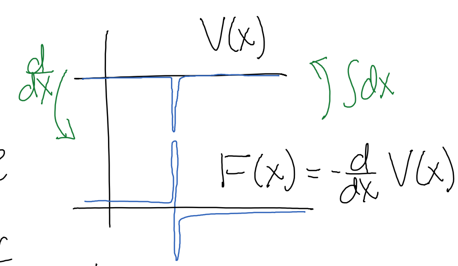
i) $\Delta\psi = 0$ ie. $\psi_1(a) = \psi_2(a)$ ii) $\Delta\psi' = 0$ ie $\psi_1'(a) = \psi_2'(a)$

If the potential has a singularity, then $\Delta\psi' \neq 0$ as in the following example:

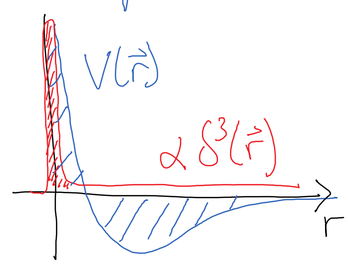
* Dirac δ-function: the "un" distribution $\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$ $\int_a^b \delta(x)\,dx = \begin{cases} 1 & \text{if } a<0<b \\ 0 & \text{otherwise} \end{cases}$

- simple definition: $\delta(x)\,dx \equiv d\,\Theta(x)$ (differential of the Heaviside step function)
- it is not a function because it's properties are defined in terms of integrals
- it is a distribution (measure, density, functional, or differential) because it "lives inside an integral" $\int dx\, \delta(x-a)\, f(x) \equiv f(a)$
- you can think of it as an "undistribution": the density function of a distribution where all the mass or charge is clumped into one spot.

* Dirac $\delta(x)$ well $V(x) = -\alpha\, \delta(x)$



- physical significance: localized infinite force
- Example: the Fermi potential captures the large-scale properties of the nuclear potential, which has a "hard core" and a finite range.



$V(r) \approx \alpha\, \delta^3(\vec{r})$ where $\alpha = \int d^3r\, V(\vec{r})$

* bound eigenstates $(E<0)$ for attractive potential $(\alpha>0)$:

TISE: $-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) - \alpha\, \delta(x)\, \psi(x) = E\, \psi(x)$

if $x \neq 0$: free particle $\frac{d}{dx}e^{\pm\kappa x} = \underbrace{\pm\kappa}_{\text{eigenvalue}}\, e^{\pm\kappa x}$ $\qquad E = -\frac{\hbar^2\kappa^2}{2m}$

general solution: $\psi_1(x) = \cancel{A}^{\,0} e^{-\kappa x} + B e^{\kappa x}$ $(x<0)$; $\qquad \psi_2(x) = F e^{-\kappa x} + \cancel{G}^{\,0} e^{\kappa x}$ $(x>0)$

$\psi_1 \to 0$ as $x \to -\infty$ $\qquad \psi_2 \to 0$ as $x \to +\infty$

external B.C.'s $\Rightarrow$ $\qquad \psi_1(x) = B e^{\kappa x}$ $\qquad \psi_2(x) = F e^{-\kappa x}$

internal B.C.'s: i) $\psi_1(0) = \psi_2(0)$: $B = F$

ii) instead of $\psi_1'(0) = \psi_2'(0)$, integrate the TISE accross the boundary:

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} dx \left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) \;+\; V(x)\psi(x) \;=\; \underbrace{E\,\psi(x)}_{0} \right]$$

$$\lim_{\varepsilon\to 0} \frac{-\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon} \cancel{dx}\, \frac{d\psi'(x)}{\cancel{dx}} = \frac{-\hbar^2}{2m}\Delta\psi'(0) = \frac{-\hbar^2}{2m}\Big(-\kappa\,\psi_2(0) - (+\kappa)\,\psi_1(0)\Big) = \frac{\hbar^2\kappa}{2m}\psi(0)$$

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} dx\, V(x)\psi(x) = \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} dx\,(-\alpha\,\delta(x))\,\psi(x) = -\alpha\,\psi(0)$$

thus, $\frac{-\hbar^2}{2m}\Delta\psi'(0) - \alpha\,\psi(0) = \left(\frac{\hbar^2\kappa}{2m} - \alpha\right)\psi(0) = 0$

and there is only one bound state: $\kappa = \frac{m\alpha}{\hbar^2}$ $\qquad E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$

Normalization: $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2\int_0^{\infty} dx\, B^2 e^{-2\kappa x} = \frac{B^2}{\kappa} = 1$ $\qquad B = \sqrt{\frac{m\alpha}{\hbar^2}}$

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}\, e^{-m\alpha|x|/\hbar^2}$$
$$E = \frac{-m\alpha^2}{2\hbar^2}$$