

* scattering eigenstates of the δ -function potential

If $E > 0$, the states will be oscillatory as $x \rightarrow \pm \infty$.

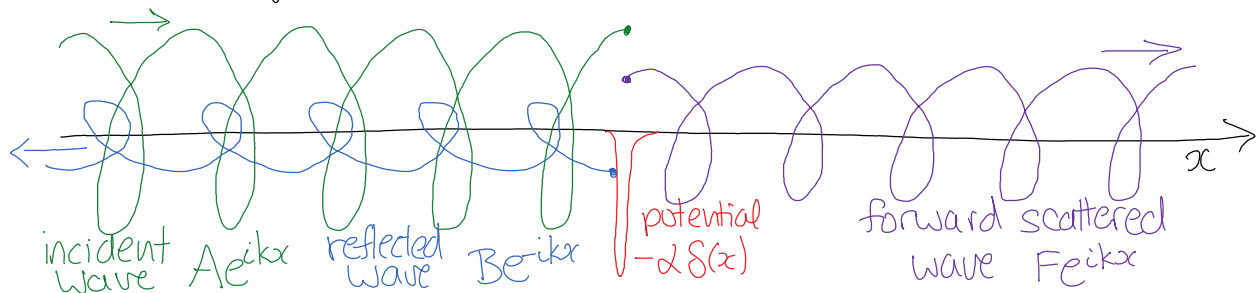
We cannot solve for discrete energy eigenstates using external B.C.'s.

There is a continuous energy spectrum (the continuum):

There is a wavefunction for every energy level.

Our attention shifts from solving for energy levels

to solving for scattering (reflection)/transition amplitudes.



+ Eigenfunctions: solve in two regions: $\psi_1(x)$ [$x < 0$], $\psi_2(x)$ [$x > 0$]

if $x \neq 0$, $\frac{d}{dx} e^{\pm i k x} = \pm i k e^{\pm i k x}$ $E = + \frac{\hbar^2 k^2}{2m}$
eigenvalue

general solution: $\psi_1(x) = \underbrace{Ae^{ikx}}_{\rightarrow} + \underbrace{Be^{-ikx}}_{\leftarrow}$ $\psi_2(x) = \underbrace{Fe^{ikx}}_{\rightarrow} + \underbrace{Ge^{-ikx}}_{\leftarrow}$

Note: $e^{\pm i k x}$ have definite momentum, while $\sin(kx) \pm i \cos(kx)$ are easier to satisfy external boundary conditions.

+ These solutions are not normalizable – continuous spectrum
 thus k is a free parameter and the 4 boundary conditions will give us the 4 parameters A, B, F, G .

* Boundary conditions:

+ External boundary conditions are different for scattering, and represent the incoming flux from the left (A) or right (G).

The flux (particles/s) is given by the probability current at $x = \pm \infty$:

$$j_c(x) = \psi^* \frac{\hat{p}}{2m} \psi - \psi \frac{\hat{p}}{2m} \psi^* = (Ce^{\pm ikx})^* \frac{-i\hbar \partial}{2m} (Ce^{\pm ikx}) + \text{c.c.} \quad [\text{complex conjugate}]$$

$$= C^* e^{\mp ikx} \cdot \frac{-i\hbar}{2m} (\pm ik) C e^{\pm ikx} + \text{same} = \pm \frac{\hbar k}{m} |C|^2 = \pm v |C|^2$$

Thus the external boundary conditions are given specifying

$$j_A(-\infty) = |A|^2 v \quad \text{and} \quad j_B(\infty) = |G|^2 v \quad [\text{usually } (A, 0) \text{ or } (0, G)].$$

+ The internal (continuity B.C.'s) are used to solve for the scattering (reflection) amplitude (B/A) or transmission amplitude (F/A). They are the same conditions as for the bound state problem:

$$\text{i) } \Delta\psi = 0 \quad [\psi_2(0) = \psi_1(0)] : \quad F + G = A + B \quad [1]$$

$$\text{ii) } \frac{-\hbar^2}{2m} \Delta\psi'(0) - \alpha\psi(0) = 0 : \quad \frac{-\hbar^2}{2m} ik((F-G)-(A-B)) = \alpha(A+B)$$

$$\text{let } \beta = \frac{m\alpha}{\hbar k} \quad \text{then} \quad F - G = (1+2i\beta)A - (1-2i\beta)B \quad [2]$$

+ Now apply the external boundary conditions and solve:

$$\begin{array}{rcl} F & A + & B \quad (1) \\ F = (1+2i\beta)A - (1-2i\beta)B & & (2) \end{array}$$

$$\begin{array}{rcl} (-) & & \\ 0 = [1-(1+2i\beta)]A + [1-(1-2i\beta)]B & & \\ (+) & & \\ [(1+2i\beta)+1]F = [(1-2i\beta)+(1+2i\beta)]A & & \end{array}$$

$$\text{Outgoing amplitude ratios: } B = \frac{i\beta}{1-i\beta} A \quad F = \frac{1}{1-i\beta} A$$

Reflection/Transmission coefficients:

$$R = \frac{j_B}{j_A} = \frac{\hbar k |B|^2}{\hbar k |A|^2} = |B/A|^2 = \left| \frac{i\beta}{1-i\beta} \right|^2 = \frac{\beta^2}{1+\beta^2}$$

$$T = \frac{j_F}{j_A} = \frac{\hbar k |F|^2}{\hbar k |A|^2} = |F/A|^2 = \left| \frac{1}{1-i\beta} \right|^2 = \frac{1}{1+\beta^2}$$

$$\text{note that } R+T = \frac{1+\beta^2}{1+\beta^2} = 1 \quad \text{No particle is lost.}$$

* Systematic approach: scattering / transfer matrices:

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combine these two eq's.
into single matrix eq:
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1+2i\beta & -1+2i\beta \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \quad \begin{matrix} [1] \\ [2] \end{matrix}$$

there are 4 unknowns and only 2 equations: we must solve for 2 coefficients in terms of 2 others.

which 2 you solve for depends on the application:

a) transfer matrix: $A, B \rightarrow F, G$: useful for "transferring" the amplitudes across multiple boundaries: $A, B \rightarrow C, D \rightarrow F, G \rightarrow \dots$

b) scattering matrix: $A, G \rightarrow B, F$ - physical coefficients for reflection & transmission from the left or right

See Griffiths 1st ed. §2.7; 2nd ed problems 2.52, 2.53

+ transfer matrix:
$$\begin{pmatrix} F \\ G \end{pmatrix} = M \begin{pmatrix} A \\ B \end{pmatrix} \quad M = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1+2i\beta & -1+2i\beta \end{pmatrix} = \begin{pmatrix} 1+i\beta & i\beta \\ -i\beta & 1-i\beta \end{pmatrix}$$

for multiple scatters, $M_{\text{tot}} = M_1 \cdot M_2$ PHY 417, HW5 #2
(recall recursive solution of cylindrical magnetic shields!)
(also recall reflection/transmission coefficients in E&M)

+ scattering matrix:
$$\begin{pmatrix} B \\ F \end{pmatrix} = S \begin{pmatrix} A \\ G \end{pmatrix} \quad S = \frac{1}{M_{22}} \begin{pmatrix} -M_{21} & 1 \\ |M| & M_{12} \end{pmatrix} = \frac{1}{1-i\beta} \begin{pmatrix} i\beta & 1 \\ 1 & i\beta \end{pmatrix}$$

Coefficients of: reflection transmission

from: left ($G=0$) $R_l \equiv \frac{|B|^2}{|A|^2} \Big|_{G=0} = |S_{11}|^2 \quad T_l \equiv \frac{k_2}{k_1} \frac{|F|^2}{|A|^2} \Big|_{G=0} = \frac{k_2}{k_1} |S_{21}|^2$

right ($A=0$) $R_r \equiv \frac{|F|^2}{|G|^2} \Big|_{A=0} = |S_{22}|^2 \quad T_r \equiv \frac{k_2}{k_1} \frac{|B|^2}{|G|^2} \Big|_{A=0} = \frac{k_2}{k_1} |S_{12}|^2$