L24-Square Well Scattering

Wednesday, November 4, 2015 08:27

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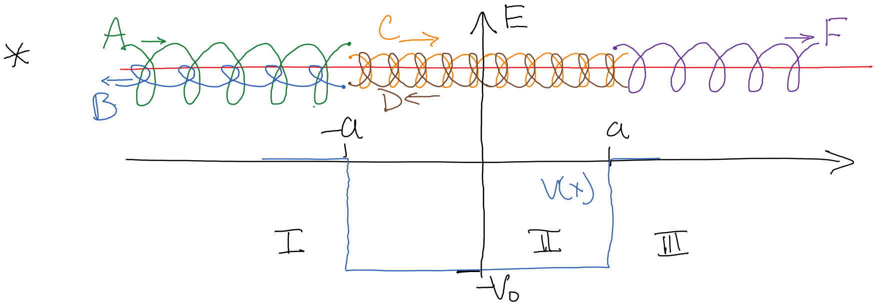


* Scattering States:

$$\psi_I = Ae^{ikx} + Be^{-ikx} \qquad E = \frac{\hbar^2 k^2}{2m}$$

$$\psi_{II} = C\sin(lx) + D\cos(lx) \qquad E + V_0 = \frac{\hbar^2 l^2}{2m}$$

$$\psi_{III} = Fe^{ikx} + Ge^{-ikx}$$

(incoming from left if $G=0$)

B.C.'s: $x=-a$ $\qquad$ $x=+a$

$$Ae^{-ika} + Be^{ika} = -C\sin(la) + D\cos(la)$$

$$ik(Ae^{-ika} - Be^{ika}) = l(C\cos(la) + D\sin(la))$$

$$C\sin(la) + D\cos(la) = Fe^{ika} + Ge^{-ika}$$

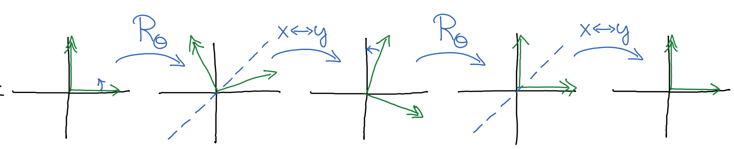
$$l(C\underbrace{\cos(la)}_{C_l} - D\underbrace{\sin(la)}_{S_l}) = ik(Fe^{ika} - Ge^{-ika})$$

$$\begin{pmatrix} 1 & 1 \\ ik/l & -ik/l \end{pmatrix}\begin{pmatrix} e^{-ika}A \\ e^{ika}B \end{pmatrix} = \begin{pmatrix} -S_l & C_l \\ C_l & S_l \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} S_l & C_l \\ C_l & -S_l \end{pmatrix}}_{R = R^{-1} = R^T\,!}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ ik/l & -ik/l \end{pmatrix}\begin{pmatrix} e^{ika}F \\ e^{-ika}G \end{pmatrix}$$

notes:

$$\begin{pmatrix} S_l & C_l \\ C_l & -S_l \end{pmatrix}^2 = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} C_l & -S_l \\ S_l & C_l \end{pmatrix}\right]^2 =$$



$$= I$$

$$\begin{pmatrix} -S_l & C_l \\ C_l & S_l \end{pmatrix}\begin{pmatrix} S_l & C_l \\ C_l & -S_l \end{pmatrix} = \begin{pmatrix} C_l & -S_l \\ S_l & C_l \end{pmatrix}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{I}\begin{pmatrix} C_l & -S_l \\ S_l & C_l \end{pmatrix} = R_l R_l = R_{2l} = \begin{pmatrix} C_{2l} & -S_{2l} \\ S_{2l} & C_{2l} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ ik/l & -ik/l \end{pmatrix}^{-1} = \left[\begin{pmatrix} 1 & \\ & ik/l \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right]^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \\ & -il/k \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -il/k \\ 1 & il/k \end{pmatrix}$$

thus

$$\begin{pmatrix} e^{-ika}A \\ e^{ika}B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -il/k \\ 1 & il/k \end{pmatrix}\underbrace{\begin{pmatrix} -S_l & C_l \\ C_l & S_l \end{pmatrix}\begin{pmatrix} S_l & C_l \\ C_l & -S_l \end{pmatrix}^{-1}}_{R_{2l}}\begin{pmatrix} 1 & 1 \\ ik/l & -ik/l \end{pmatrix}\begin{pmatrix} e^{ika}F \\ e^{-ika}G \end{pmatrix}$$

$$= \begin{pmatrix} C_{2l} - i\frac{k^2+l^2}{2kl}S_{2l} & i\frac{k^2-l^2}{2kl}S_{2l} \\ -i\frac{k^2-l^2}{2kl}S_{2l} & C_{2l} + i\frac{k^2+l^2}{2kl}S_{2l} \end{pmatrix}\begin{pmatrix} e^{ika}F \\ e^{-ika}G \end{pmatrix}$$

$$= \begin{pmatrix} C_{2\ell} - i\frac{k^2+\ell^2}{2k\ell}S_{2\ell} & i\frac{k^2-\ell^2}{2k\ell}S_{2\ell} \\ -i\frac{k^2-\ell^2}{2k\ell}S_{2\ell} & C_{2\ell}+i\frac{k^2+\ell^2}{2k\ell}S_{2\ell} \end{pmatrix} \begin{pmatrix} e^{ika}F \\ e^{-ika}G \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \left(C_{2\ell} - i\frac{k^2+\ell^2}{2k\ell}S_{2\ell}\right)e^{2ika} & -i\frac{k^2-\ell^2}{2k\ell}S_{2\ell} \\ -i\frac{k^2-\ell^2}{2k\ell}S_{2\ell} & \left(C_{2\ell}+i\frac{k^2+\ell^2}{2k\ell}S_{2\ell}\right)e^{-2ika} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = M \begin{pmatrix} F \\ G \end{pmatrix}$$

$$\begin{pmatrix} B \\ F \end{pmatrix} = \frac{1}{M_{11}} \begin{pmatrix} M_{21} & 1 \\ 1 & -M_{12} \end{pmatrix} \begin{pmatrix} A \\ F \end{pmatrix} = S \begin{pmatrix} A \\ F \end{pmatrix}$$

$$\left|\frac{k^2+\ell^2}{2k\ell}\right|^2 = 1 + \left|\frac{k^2-\ell^2}{2k\ell}\right|^2$$

$$\det M = 1$$

$$T_\ell = \frac{|F|^2}{|A|^2} = |S_{21}|^2 = |M_{11}|^{-2} = \left[C_{2\ell}^2 + \left(\frac{k^2+\ell^2}{2k\ell}\right)^2 S_{2\ell}^2\right]^{-1} = \left[1 + \left(\frac{k^2-\ell^2}{2k\ell}\right)^2 S_{2\ell}^2\right]^{-1}$$

$$R_\ell = \frac{|B|^2}{|A|^2} = |S_{11}|^2 = |M_{21}|^2 T_\ell = \underbrace{\left(\frac{k^2-\ell^2}{2k\ell}\right)^2 S_{2\ell}^2}_{X} T_\ell = \left[\left(\left(\frac{k^2-\ell^2}{2k\ell}\right)^2 S_{2\ell}^2\right)^{-1} + 1\right]^{-1}$$

$$T_\ell = \left[1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)\right]^{-1} = \left[1 + \frac{z_0^4}{4(z^2-z_0^2)z^2}\sin^2 z\right]^{-1}$$

$$T_\ell + R_\ell = \frac{1}{1+X} + \frac{X}{1-X} = 1$$

where $(z \equiv 2\ell a)^2 = \frac{8ma^2(E+V_0)}{\hbar^2}$ $\quad z_0^2 \equiv (2\ell a)^2 - (2ka)^2 = \frac{8ma^2V_0}{\hbar^2}$

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Plot[
 Evaluate[Join[Table[1 / (1 + z0^4 / (4 * (z^2 - z0^2) * z^2) * Sin[z]^2) * UnitStep[z - z0],
   {z0, {0, Pi / 4, Pi / 2, 3 * Pi / 4, Pi, 5 * Pi / 4, 5 * Pi / 2, 9 * Pi / 2}}], {Sin[z]^2 / 2}]], {z, 0, 25}]
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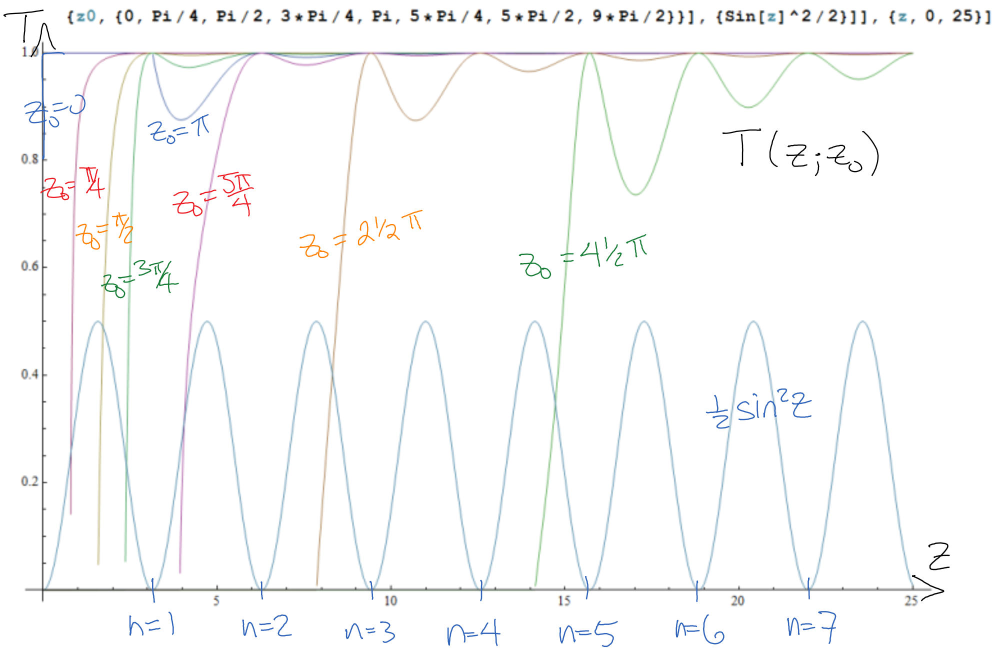

* resonances carry the same B.C.'s from $-a \to a$ and have 100% transmission