L25-Vectors, Duals, Metric, Adjoint

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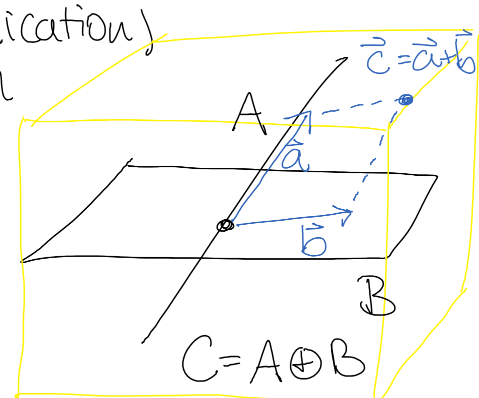
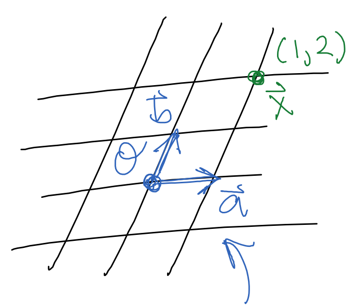
* vectors
  - prototype: arrows add head-to-tail, stretch
    column vectors of components
  - formal: objects with linear combinations
    $\alpha \vec{u} + \beta \vec{v} + \dots = \sum \alpha_i \vec{v}_i \rightarrow \alpha_i \vec{v}_i$ (index notation: implied summation)
    - properties of scalar product, addition ensure this
  - strange examples:
    - apples & bananas & cherries $i\vec{a} + j\vec{b} + k\vec{c}$
    - functions: $(\alpha_i f_i)(x) = \alpha_i \cdot f_i(x)$ notation: $\langle x | \boxed{\alpha_i | f_i\rangle}$
    - complex numbers: $z = x + iy$ basis: $1, i$
    - matrices: (without matrix multiplication)
  - subspaces: closed under addition
    - subplanes, sublines, etc, all through the origin
    - direct sum of subspaces $C = A \oplus B$
      $\vec{c} = \vec{a} + \vec{b} \sim (\vec{a}, \vec{b})$
    - $\vec{a}, \vec{b}$ are the projection of $\vec{c}$ into these subspaces (not orthogonal! no metric)



  - basis: break space down to line subspaces (1-d) and pick one vector on each line.
    - projection to components (α, β, γ)
      $\vec{x} = \vec{a}\,\alpha + \vec{b}\,\beta + \vec{c}\,\gamma$
      $= \underbrace{(\vec{a}\ \vec{b}\ \vec{c})}_{\text{basis}} \underbrace{\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}}_{\text{components}}$



    - structure of any vector space is characterized by its basis + eggcrate extension
    - a basis must be linearly independent and complete

  - what does this buy us for Quantum Mechanics?
    superposition / interference of complex amplitudes

* inner product – metric, contraction
  – adding structure to vector space as "multilinear extensions"

$$\vec{a}\cdot\vec{b} = a_x^* b_x + a_y^* b_y + \dots = (a_x^* a_y^* \dots)\begin{pmatrix} b_x \\ b_y \\ \vdots \end{pmatrix} = \mathbb{a}^{T*}\mathbb{b}$$

$$\bullet \equiv \; ^{T*} \equiv \; ^{\dagger}$$

– properties:
  a) multilinear $\quad (\alpha_i \vec{a}_i)\cdot\vec{b} = \alpha_i(\vec{a}_i\cdot\vec{b})$, also for $\beta_i \vec{b}_i$
  b) symmetric $\quad \vec{a}\cdot\vec{b} = (\vec{b}\cdot\vec{a})^*$ (almost!)
  c) scalar valued – contracts two vectors to a scalar.

– vs outer product:
$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}(b_x b_y b_z) = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix} \quad \text{Trace.}$$
  · the trace of the outer product = inner product

– inner product of functions (as vectors):

$$\vec{a}\cdot\vec{b} = \sum_i a_i^* b_i \rightarrow \langle f|g\rangle = \int dx\, f^*(x)\, g(x)$$
$$\uparrow \text{index } i \qquad\qquad \uparrow \text{index } x.$$

  · we will continue this extended analogy next week

– generic metric: $\vec{x}\cdot\vec{y} = (\vec{b}_i x^i)\cdot(\vec{b}_j x^j)$
$$\vec{x}\cdot\vec{y} = (\mathbb{b}\,\mathbb{x})^{T}\cdot(\mathbb{b}\,\mathbb{y}) = \mathbb{x}^T \underbrace{\mathbb{b}^T\cdot\mathbb{b}}_{G \leftarrow \text{metric}} \mathbb{y} = (x^1 x^2 x^3)\begin{pmatrix} \vec{b}_1\cdot\vec{b}_1 & \vec{b}_1\cdot\vec{b}_2 & \vec{b}_1\cdot\vec{b}_3 \\ \vec{b}_2\cdot\vec{b}_1 & \vec{b}_2\cdot\vec{b}_2 & \vec{b}_2\cdot\vec{b}_3 \\ \vec{b}_3\cdot\vec{b}_1 & \vec{b}_3\cdot\vec{b}_2 & \vec{b}_3\cdot\vec{b}_3 \end{pmatrix}\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}$$
$$= \mathbb{x}^T G\, \mathbb{y} = g_{ij} x^i y^j$$
  · a symmetric matrix in general.

– orthonormal basis: $\boxed{\hat{e}_i\cdot\hat{e}_j = \delta_{ij}} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$
  (implies independence)
  · usefull for picking out components
$$\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix}\cdot(\hat{e}_1 \hat{e}_2 \hat{e}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}$$
$$\hat{\mathbb{e}}\cdot\vec{V} = \hat{\mathbb{e}}\cdot(\hat{\mathbb{e}}\,\mathbb{v}) = (\hat{\mathbb{e}}\cdot\hat{\mathbb{e}})\mathbb{v} = \mathbb{I}\,\mathbb{v} = \mathbb{v}$$

- closure relationship $\vec{V} = \hat{e}\mathbb{V} = \underbrace{\hat{e}\,\hat{e}\cdot}_{1 = \text{identity on } V}\vec{V} = 1\,\vec{V}$

$\hat{e}\,\hat{e}\cdot = 1$ or $\boxed{\sum_i \hat{e}_i\,\hat{e}_i\cdot = 1}$

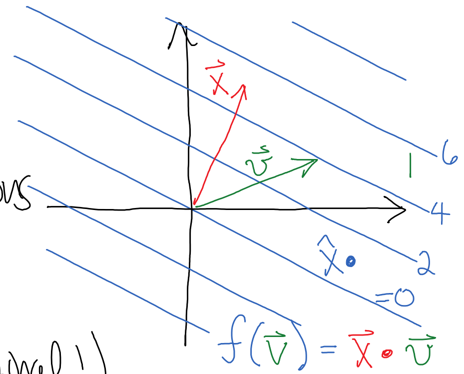
(implies completeness).

- orthogonal projections $\boxed{P_i = \hat{e}_i\,\hat{e}_i\cdot}$ note the $\cdot$!

$P_x = \hat{x}\,\hat{x}\cdot \sim \begin{pmatrix}1\\0\\0\end{pmatrix}(1\,0\,0) = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $\quad P_x^2 = P_x \quad P_x + P_y + P_z = I$

* Linear functionals (Forms): Dual Space
(half of the inner product).

- the object $\vec{x}\cdot \sim (x_1^* \; x_2^* \ldots)$
is a linear function of $\vec{v}$
- it is a vector itself under operations
$[\alpha(\vec{x}_1\cdot) + \beta(\vec{x}_2\cdot)](\vec{y}) \equiv \alpha\,\vec{x}_1\cdot\vec{y} + \beta\,\vec{x}_2\cdot\vec{y}$
- this is called the dual space.
(the dual of the dual is $\sim$ the original!)



- the cobasis is defined to be "orthonormal"
$\boxed{\tilde{e}_i(\vec{e}_j) \equiv \delta_{ij}}$ (can be defined even without a metric)

- can be used to extract co-ordinates:

$$\tilde{e}_i(\vec{x}) = \tilde{e}_i(\alpha_j\,\vec{e}_j) = \alpha_j\,\tilde{e}_i(\vec{e}_j) = \alpha_j\,\delta_{ij} = \alpha_i$$

$$\tilde{x}(\vec{e}_i) = (\beta_j\,\tilde{e}_j)(\vec{e}_i) = \beta_j\cdot\tilde{e}_j(\vec{e}_i) = \beta_j\,\delta_{ji} = \beta_i$$

* Adjoint: the metric provides a mapping between a vector space and it's adjoint.

$$\vec{V} \longrightarrow \tilde{V} \equiv \vec{V}\cdot \equiv \vec{V}^{\dagger} \qquad \tilde{V}(\vec{x}) = \vec{V}\cdot\vec{x}$$

- for orthonormal basis:

$\tilde{e}_i = \hat{e}_i \cdot$ since $\tilde{e}_i(\vec{e}_j) = \delta_{ij} = \hat{e}_i \cdot \hat{e}_j$

- thus $\vec{V}^\dagger = (\hat{e}_i V^i)^\dagger = (\hat{e}_i V_i)\cdot = V_i^* \hat{e}_i \cdot = V_i^* \tilde{e}_i$

  The adjoint of a vector is the conjugate transpose.
  row vectors are waiting to gobble up colum vectors.

- same for functions: $\langle f | g \rangle = \int dx\, f^*(x)\, g(x)$

  bra: $\langle f | \sim \int dx\, f^*(x)$ a linear functional of functions
  ket: $\underbrace{|g\rangle}_{\text{vector \& dual}} \sim \underbrace{g(x)}_{x^{th}\text{ component.}}$ a plain old function

- note that $\tilde{f}\vec{g}$ or $\langle f | g \rangle$ is an inner product
  $\vec{f}\tilde{g}$ or $|f\rangle\langle g|$ is an outer product.