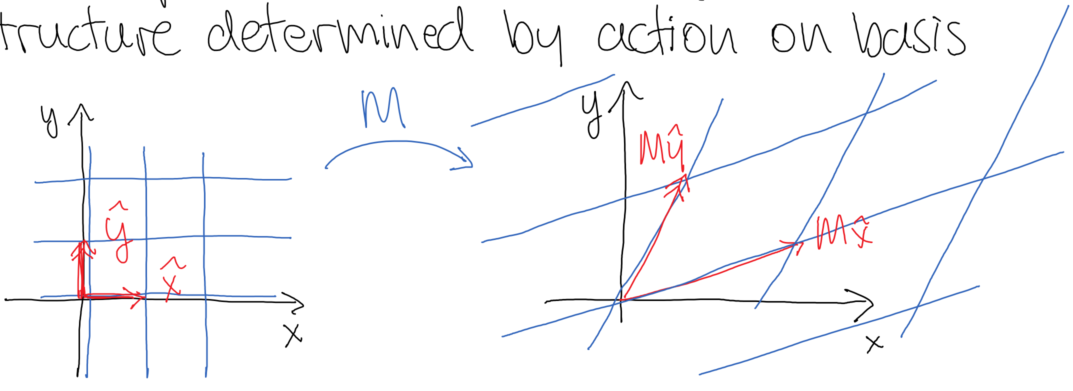
- * linear operators - functions from V → V
  - structure determined by action on basis



- * components

$$\vec{w} = M(\vec{v}) = M(\vec{\mathbf{b}}\,\mathbf{v})$$

$$\vec{\mathbf{b}}\,\mathbf{w} = M(\vec{\mathbf{b}})\,\mathbf{v} = \vec{\mathbf{b}}\,\mathbf{M}\,\mathbf{v}$$

$$\mathbf{w} = \mathbf{M}\,\mathbf{v} \quad \text{where} \quad \boxed{\mathbf{M} = \tilde{\mathbf{b}} \cdot M(\vec{\mathbf{b}})}$$

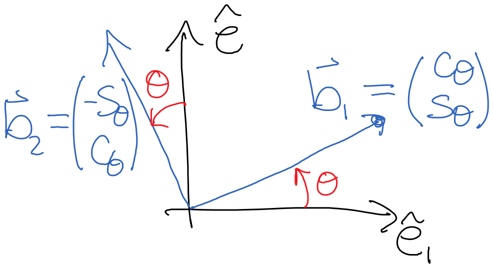
$$\underbrace{\tilde{\mathbf{b}}(\vec{w})}_{\mathbf{w}} = \overbrace{\underbrace{\tilde{\mathbf{b}}(M(\vec{\mathbf{b}}))}_{\mathbf{M}}\,\underbrace{\tilde{\mathbf{b}}(\vec{v})}_{\mathbf{v}}}^{\text{closure}}$$

$$M_{ij} = \hat{e}_i \cdot M(\hat{e}_j) \sim \langle x|A|y\rangle$$

(matrix elements)

matrix operators inherit 2 bases:
from domain & range. !

- * Rotations (active)



let $\vec{b}_1 = R\hat{e}_1$ $\quad \vec{b}_2 = R\hat{e}_2$

combine: $(\vec{b}_1\ \vec{b}_2) = R(\hat{e}_1\ \hat{e}_2)$

$$\vec{\mathbf{b}} = R\hat{\mathbf{e}}$$

$$\begin{pmatrix} b_1^x & b_2^x \\ b_1^y & b_2^y \end{pmatrix} = \begin{pmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- rotations are "orthogonal" (preserve the metric)

$$\vec{\mathbf{b}}^T \cdot \vec{\mathbf{b}} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix} \cdot (\vec{b}_1\ \vec{b}_2) = \mathbb{I} \qquad R^T R = \mathbb{I}$$

$$\vec{b}_i \cdot \vec{b}_j = \delta_{ij}$$

in general, $R^{adj} \cdot R = \mathbb{I}$, or $\boxed{R^\dagger G R = G}$

* Hermitian adjoint of a operator (complex transpose)

- how an operator commutes with the metric

$$\mathbb{M}(\vec{a})\cdot\vec{b} \equiv \vec{a}\cdot\mathbb{M}^{adj}(\vec{b}) \qquad \langle Mf|g\rangle = \langle f|M^{\dagger}g\rangle$$

$$(\mathbb{M}\,\mathbb{a})^{\dagger}\mathbb{G}\,\mathbb{b} = \mathbb{a}^{\dagger}\mathbb{G}\,\mathbb{M}^{adj}\mathbb{b}$$

$$\mathbb{M}^{\dagger}\mathbb{G} = \mathbb{G}\,\mathbb{M}^{adj} \qquad \boxed{\mathbb{M}^{adj} = \mathbb{G}^{-1}\mathbb{M}^{\dagger}\mathbb{G}} \rightarrow \enclose{circle}{\mathbb{M}^{\dagger}} \text{ if } \mathbb{G}=\mathbb{I}$$

- a matrix multiplies "to the left" as its adjoint

if $\mathbb{b} = \mathbb{a}\mathbb{M}$ $\quad (b_1\, b_2) = (a_1\, a_2)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ $\quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}^{*} = \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}^{*}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{*}$

then $\mathbb{b}^{\dagger} = \mathbb{M}^{\dagger}\mathbb{a}^{\dagger}$

note the transpose

- Adjoint is a duality: $(\mathbb{M}^{adj})^{adj} = \mathbb{M}$

- Adjoint of product: $(A\cdot B)^{adj} = B^{adj}\cdot A^{adj}$ like $(AB)^{-1} = B^{-1}A^{-1}$

- Unified operation (†) for adjoint of operators, vectors, scalars!

* Active vs. Passive rotations

- Active rotations: physically move vectors
  - only one basis, $\vec{b}_i$ are just rotated vectors.

$$\vec{v}' = R\vec{v} \qquad \vec{v} = \hat{x}v_x + \hat{y}v_y \rightarrow \vec{v}' = \hat{x}v_x' + \hat{y}v_y'$$

  - example: evolution of the wavefunction in time

- Passive rotations (identity transform) $\vec{v}$ stays put
  - turn your head and look at it from different angle.
  - new components correspond to a change of basis

$$\vec{v} = \overbrace{\vec{\mathbb{b}}}\,\underbrace{\mathbb{v}'} = \underbrace{\overbrace{\hat{e}\,\mathbb{R}}\,\mathbb{v}} = \hat{e}\,\mathbb{v} = \vec{v} \quad \text{(still)}$$

$$\vec{\mathbb{b}} = \hat{e}\,\mathbb{R} \quad\Rightarrow\quad \mathbb{v}' = \mathbb{R}\,\mathbb{v}$$

$$\underbrace{\vec{v} = 1\,\vec{v}}_{\text{identity transform}} \qquad \underbrace{\vec{b}^{T}\cdot\vec{v}}_{\mathbb{v}'} = \underbrace{\vec{b}^{T}\cdot 1\,\hat{e}}_{\mathbb{R}}\overset{\text{closure}}{\vphantom{x}}\underbrace{\hat{e}^{T}\cdot\vec{v}}_{\mathbb{v}}$$

(wait for L13)

- example: Fourier transform: $\langle x|f\rangle = \int dk \langle x|k\rangle\langle k|f\rangle$

– Similarity transformation: change of basis of a matrix

if $\mathbb{M} = \hat{e}^T \cdot M(\hat{e})$ are matrix elements in the basis $\hat{e}$,

then $\mathbb{M}' = \vec{b}^{-1} \cdot M(\vec{b}) = \underbrace{\vec{b}^{-1} \cdot \hat{e}}_{\mathbb{B}^{-1}} \underbrace{\hat{e}^T \cdot M(\hat{e}}_{\mathbb{M}} \underbrace{\hat{e}^T \cdot \vec{b}}_{\mathbb{B}})$ in the basis $\vec{b}$

thus $\boxed{\mathbb{M}' = \mathbb{B}^{-1} M \mathbb{B}}$ change-of-basis formula for matrices

$\mathbb{B} = \hat{e}^T \cdot 1 \vec{b}$ transforms the domain $(\vec{v})$
$\mathbb{B}^{-1} = \vec{b}^{-1} \cdot 1 \hat{e}$ transforms the range $(\vec{w})$